A new efficient algorithm to compute the exact reflection and transmission factors for plane waves in layered absorbing media (liquids and solids)

Pierre Cervenka and Pascal Challande
Laboratoire de Mécanique Physique, 2, place de la gare de ceinture, 78210 Saint-Cyr-l'Ecole France

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This paper describes a matrix method for computing the exact reflection and transmission coefficients for harmonic plane waves within a stratified medium of homogeneous, isotropic, and absorbing plane layers. The new feature is that each layer can be either liquid or solid, whatever their successive order. Furthermore, this algorithm takes into account evanescent waves, but also applies whatever the thickness of each layer. A numerical example is shown.

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INTRODUCTION

The transmission and reflection of acoustic waves in layered media has been extensively studied. An early paper by Thompson[1] (corrected by Haskell[2]) and two reference books[3,4] can be cited. Many other works have followed (e.g., Refs. 5–10 include large bibliographies), but none of them address the case of layered media with any sequence of liquid and solid layers. However, this issue is of a practical interest within different domains, e.g., nondestructive testing. The present paper is concerned with plane layers stacked between two semi-infinite media, one of which contains an incident wave and a reflected wave. This region will be called the input. The other semi-infinite medium contains only a transmitted wave. Hence, it will be called the output. Each layer and the surrounding as well, are homogeneous, isotropic, and attenuate acoustic waves. They are liquids or solids of any thickness. Accordingly, the solution given here may apply to a large range of situations.

It is known that a harmonic radiation propagating within a homogeneous medium can be analyzed with Fourier's formalism as a superposition of inhomogeneous plane modes (Alais[11]). These modes have a constant amplitude in every plane parallel to a reference plane. By choosing the planes in which the decomposition takes place so that they are parallel to the interfaces between the layers, this invariance remains when propagating across the stratified medium. In studying the reflection and transmission of a harmonic beam, the elementary problem reduces to consideration of such modes sharing the same wave vector projection **K** onto these planes ($|\mathbf{K}|$ is a real number). Mackenzie[12] made implicit use of these modes to deal with the sea-bottom interface, and the spatial Fourier analysis has been the basis of other related studies (e.g., Ref. 13) since then. The model described here is intended to calculate the complex elements of the (4×2) matrix, M_K, related to a given projection **K** so that

$$\begin{bmatrix} \Phi \\ a_r^L \\ \Psi \\ a_r^S \end{bmatrix} = M_K \begin{bmatrix} a_i^L \\ a_i^S \end{bmatrix}, \tag{1}$$

in which (a^L, Φ) and (a^S, Ψ) denote, respectively, the complex amplitudes of longitudinal and shear waves in terms of the velocity potential. The quantities a_i and a_r are the amplitudes of the incident and reflected waves at the first interface, in the "input" medium, whereas Φ and Ψ refer to the transmitted waves at the last interface in the "output" semi-infinite medium. The shear wave's amplitudes (a^S, Ψ) and the related coefficients in M_K are indeed only used when relevant, i.e., when the input and/or output medium is a solid. It must be noticed that the present paper only deals with the SV shear waves for which the wave vectors lie within the plane of incidence: the SH shear waves are never converted into longitudinal or SV shear waves and conversely, so that the independent problem of the SH wave propagation is not considered here.

The theoretical model used here is exact; that is to say particularly that evanescent waves, i.e., with a large imaginary coefficient of attenuation versus a small propagating real part of the wave vector, which may produce a significant effect across thin layers, are taken into account. The way of writing the equations follows that of Brekhovskikh[3] and is based upon handling matrices that transfer velocities and stresses between interfaces. First, the matrices that are related to solids (Sec. I) and liquids (Sec. II) are recalled. This paper is mainly devoted to providing a method which can treat any sequence of layers and overcomes numerical problems that arise for certain cases. The model used to describe a solid layer in which an evanescent longitudinal wave propagates is emphasized in Sec. III, and specific results with $\mathbf{K} = \mathbf{0}$ are given in Sec. IV as well. The final stacking of the matrices which provides the expected results is described in Sec. V. A numerical example is shown in Sec. VI.

I. SOLID MEDIA

A. Elementary transfer matrix

Let us denote ϕ the scalar- and ψ the vectorial- potential fields, respectively related to the longitudinal and shear waves that propagate in a solid, and from which the acoustic velocity, **v**, is derived:

$$\mathbf{v} = \text{grad}(\phi) + \text{rot}(\psi) \tag{2}$$

The stress tensor, σ, is related to the strain rate tensor, ϵ, by

$$\sigma = 2\mu\epsilon + \lambda\theta\,\mathbf{1}, \tag{3}$$

in which θ denotes the trace of the tensor ϵ, $\mathbf{1}$ is the unit tensor, λ and μ are the Lamé coefficients. We consider only harmonic (angular frequency ω) plane longitudinal and SV shear waves of which the wave vector and the velocity vector lie in the same vertical plane. This plane is defined by the axis $0x$ (parallel to the interface) and $0z$ (normal to the interface). Thus the geometry of the problem is reduced to two dimensions and the only nonzero components derived from Eqs. (2) and (3) are written:

$$v_x = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}, \quad v_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}, \tag{4a}$$

(ψ denotes here the component ψ_y)

$$\sigma_{xz} = T_x = j\frac{\mu}{\omega}\left\{\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right\}$$

$$= j\frac{\mu}{\omega}\left\{2\,\frac{\partial^2\phi}{\partial x\,\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right\}, \tag{4b}$$

$$\sigma_{zz} = T_z = \frac{j}{\omega}\left\{2\mu\,\frac{\partial v_z}{\partial z} + \lambda\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right]\right\}$$

$$= \frac{j}{\omega}\left\{(2\mu+\lambda)\,\frac{\partial^2\phi}{\partial z^2} + \lambda\,\frac{\partial^2\phi}{\partial x^2} + 2\mu\,\frac{\partial^2\psi}{\partial x\,\partial z}\right\}. \tag{4c}$$

The volume and shear viscosities of the absorbing solid are taken into account by using complex Lamé coefficients. They are related to the complex squares of the wave vectors $\mathbf{k}^L$ (longitudinal) and $\mathbf{k}^S$ (shear) by

$$(\mathbf{k}^L)^2 = \rho\omega^2/(\lambda + 2\mu) \quad \text{and} \quad (\mathbf{k}^S)^2 = \rho\omega^2/\mu, \tag{5}$$

in which ρ denotes the specific mass of the solid.

Considering a given type of wave (longitudinal or shear), there are two waves whose projection of the wave vector $\mathbf{k}$ onto a plane Π_z parallel to the interfaces is $\mathbf{K}$: one which propagates forward (increasing z) and whose amplitude is denoted a_+; the other propagating backward with the amplitude denoted a_- (Fig. 1). The projection $\mathbf{k}_z$ of $\mathbf{k}$ onto the axis $0z$ is simply related to $\mathbf{K}$ with

$$\mathbf{k}^2 = \mathbf{k}_z^2 + \mathbf{K}^2, \tag{6}$$

in which $\mathbf{k}^2$ is given by expressions (5). The only physical solution is easily shown to be written $k_z = k_r + j\alpha$ with k_r, $\alpha \in \mathbb{R}^+$. Using this convention and for a given real vector $\mathbf{K}$, the potentials ϕ and ψ in the plane Π_z which intersects the axis $0z$ at the abscissa z are finally written:

$$\phi(z,t) = [a_+^L \exp(jk_z^L z) + a_-^L \exp(-jk_z^L z)]\,e^{j(Kx-\omega t)}, \tag{7a}$$

$$\psi(z,t) = [a_+^S \exp(jk_z^S z) + a_-^S \exp(-jk_z^S z)]\,e^{j(Kx-\omega t)}. \tag{7b}$$

The sum and difference notations are also defined (with the subscript u standing for L or S):

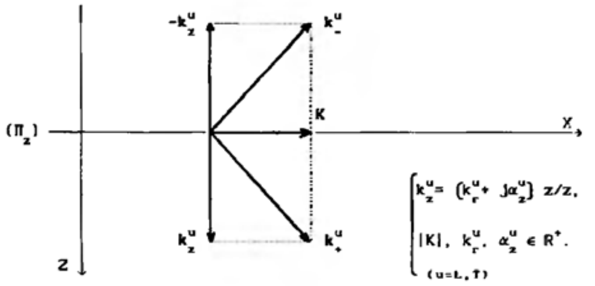

FIG. 1. Definition of wave vector components.

$$\mathscr{S}^u(z) = a_+^u \exp(jk_z^u z) + a_-^u \exp(-jk_z^u z), \tag{8a}$$

$$\mathscr{D}^u(z) = a_+^u \exp(jk_z^u z) - a_-^u \exp(-jk_z^u z), \tag{8b}$$

and the matrices of propagation:

$$\mathscr{P}^u(z) = \begin{bmatrix} C^u(z) & S^u(z) \\ S^u(z) & C^u(z) \end{bmatrix}, \tag{9}$$

with

$$\begin{cases} C^u(z) = \cosh(jk_z^u z) = \cos(k_z^u z), \\ S^u(z) = \sinh(jk_z^u z) = j\sin(k_z^u z), \end{cases}$$

that give the transfer relation for $\mathscr{S}^u$ and $\mathscr{D}^u$ from Π_z to Π_{z+d} inside the same solid layer:

$$\begin{bmatrix} \mathscr{S}^u \\ \mathscr{D}^u \end{bmatrix}_{z+d} = \mathscr{P}^u(d)\begin{bmatrix} \mathscr{S}^u \\ \mathscr{D}^u \end{bmatrix}_z$$

or also

$$\begin{bmatrix} \mathscr{D}^u \\ \mathscr{S}^u \end{bmatrix}_{z+d} = \mathscr{P}^u(d)\begin{bmatrix} \mathscr{D}^u \\ \mathscr{S}^u \end{bmatrix}_z. \tag{10}$$

From Eqs. (7a) and (7b) and with the notations (8a) and (8b), the relations (4a)–(4c) take the following matrix forms:

$$\begin{bmatrix} v_x \\ \omega T_z \end{bmatrix}_z = \mathscr{M}_a \begin{bmatrix} \mathscr{S}^L \\ \mathscr{D}^S \end{bmatrix}_z j e^{j(Kx-\omega t)}$$

and

$$\begin{bmatrix} v_z \\ \omega T_x \end{bmatrix}_z = \mathscr{M}_b \begin{bmatrix} \mathscr{D}^L \\ \mathscr{S}^S \end{bmatrix}_z j e^{j(Kx-\omega t)}, \tag{11}$$

with

$$\mathscr{M}_a = \begin{bmatrix} K & -k_z^S \\ -\rho\omega^2 C_2 & -2\rho\omega^2 k_z^S S/k^S \end{bmatrix}$$

and

$$\mathscr{M}_b = \begin{bmatrix} k_z^L & K \\ -2\rho\omega^2 k_z^L S/k^S & \rho\omega^2 C_2 \end{bmatrix},$$

in which the practical notations $S = K/k^S$, $C_2 = 1 - 2S^2$ appear.

By combining the (2×2) matrices $\mathscr{M}_a$ and $\mathscr{M}_b$, the sparse (4×4) matrix $\mathscr{M}$ is built so that:

$$\begin{bmatrix} v_x \\ v_z \\ \omega T_z \\ \omega T_x \end{bmatrix}_z = \mathscr{M}\begin{bmatrix} \mathscr{S}^L \\ \mathscr{D}^L \\ \mathscr{D}^S \\ \mathscr{S}^S \end{bmatrix}_z j e^{j(Kx-\omega t)}, \tag{12}$$

whereas the inverse relation is easily obtained for $\mathscr{M}^{-1}$ by combining $\mathscr{M}_a^{-1}$ and $\mathscr{M}_b^{-1}$. The matrices $\mathscr{P}^L(d)$ and $\mathscr{P}^S(d)$ (Eq. 9) are also merged to make, using Eqs. (10), the (4×4) matrix $\mathscr{P}(d)$ such that

$$\begin{bmatrix} \mathscr{S}^L \\ \mathscr{D}^L \\ \mathscr{D}^S \\ \mathscr{S}^S \end{bmatrix}_{z+d} = \mathscr{P}(d) \begin{bmatrix} \mathscr{S}^L \\ \mathscr{D}^L \\ \mathscr{D}^S \\ \mathscr{S}^S \end{bmatrix}_{z},$$

$$\text{with } \mathscr{P}(d) = \begin{bmatrix} \mathscr{P}^L(d) & 0 \\ 0 & \mathscr{P}^S(d) \end{bmatrix} \quad (13)$$

[0 is the (2×2) null matrix].

In a solid layer, the components (**v**,**T**) are finally transferred from Π_z to Π_{z+d} with the (4×4) matrix $a(d)$ derived from Eqs. (12) and (13):

$$\begin{bmatrix} v_x \\ v_z \\ \omega T_z \\ \omega T_x \end{bmatrix}_{z+d} = a(d) \begin{bmatrix} v_x \\ v_z \\ \omega T_z \\ \omega T_x \end{bmatrix}_{z},$$

$$\text{with } \boxed{a(d) = \mathscr{M} \times \mathscr{P}(d) \times \mathscr{M}^{-1}} \quad (14)$$

[$\mathscr{M}$, $\mathscr{M}^{-1}$ and $a(d)$ are explicitly written in the Appendix].

From a practical point of view, it can be noticed that $a(d)$ is symmetrical about its second diagonal ($a_{i,j} = a_{5-j,5-i}$). On the other hand, these transfer matrices are well suited for numerical analysis as it can be seen that $|\mathscr{P}(d)| = 1$, so that $|a(d)| = 1$ as well. A recent study by Scharnhorst[14] is dedicated to such properties of the transfer matrices in layered media.

The elementary matrix a can be used straightforwardly as far as the plane (longitudinal or shear) waves actually propagate. If the value of $|\mathbf{K}|$ is significantly greater than the critical value ω/c^L (e.g., when a wave occurs from the transmission of an incident wave at a supercritical angle), the longitudinal waves (a_+^L and a_-^L) are then highly attenuated (hence, so-called evanescent). The elements of the partial matrix $\mathscr{P}^L(d)$ grow exponentially with d, and the matrix a becomes numerically singular. Physically, it reflects the fact that an evanescent wave cannot propagate across a thick enough layer: there is not a 4×4 relation between planes Π_z and Π_{z+d} as in Eq. (14), whatever the computing precision used for handling the problem. Similar problems arise with a normally propagating wave ($\mathbf{K} = \mathbf{0}$). These cases are clarified in detail in Secs. III and IV.

B. Stack of layers

Let us consider n contiguous solid layers (Fig. 2), with tilted modes ($\mathbf{K} \neq \mathbf{0}$) which do not induce the previously mentioned evanescent propagation in any layer. The thickness of each layer is denoted d_i ($i = 1,n$). Every layer is then described by a matrix $a_i(d_i)$. The abscissa of the interface between the media (i) and ($i+1$) is denoted z_i so that $d_i = z_i - z_{i-1}$. The media (0) and ($n+1$) may be either liquid or solid. The following continuity relations hold on solid–solid interfaces:

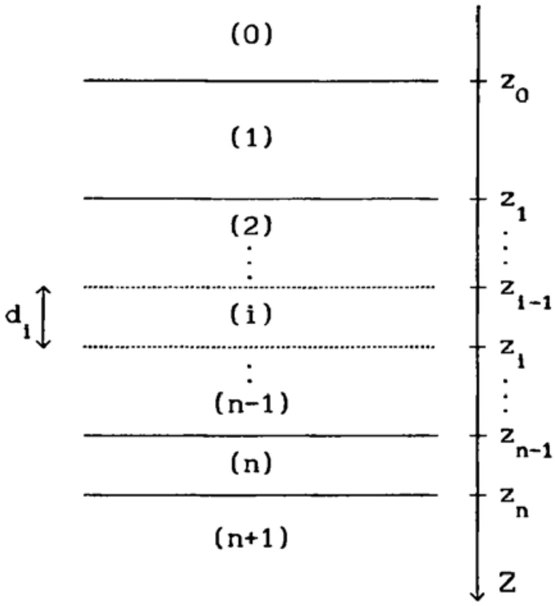

FIG. 2. Schematic diagram of n contigous solid layers.

$$\begin{bmatrix} v_x^{(j+1)} \\ v_z^{(j+1)} \\ \omega T_z^{(j+1)} \\ \omega T_x^{(j+1)} \end{bmatrix}_{z_j} = \begin{bmatrix} v_x^{(j)} \\ v_z^{(j)} \\ \omega T_z^{(j)} \\ \omega T_x^{(j)} \end{bmatrix}_{z_j}. \quad (15)$$

If one of the external media (0) and/or ($n+1$) has a solid–liquid interface with the stack, only v_z, T_z, and T_x satisfy the continuity equation (here with $z = z_0$ or $z = z_n$), T_x being null. In every case, the stack of solid layers can be described with the global (4×4) matrix $\mathscr{A}$:

$$\begin{bmatrix} v_x^{(n)} \\ v_z^{(n+1)} \\ \omega T_z^{(n+1)} \\ \omega T_x^{(n+1)} \end{bmatrix}_{z_n} = \mathscr{A} \begin{bmatrix} v_x^{(1)} \\ v_z^{(0)} \\ \omega T_z^{(0)} \\ \omega T_x^{(0)} \end{bmatrix}_{z_0},$$

$$\text{with } \boxed{\mathscr{A} = a^{(n)}(d_n) \times \cdots \times a^{(1)}(d_1)}$$

$$(|\mathscr{A}| = 1 \text{ is still true}). \quad (16)$$

But if the medium ($n+1$) is a solid [resp. medium (0)], the quantity $v_x^{(n+1)}$ (resp. $v_x^{(0)}$) can be used in place of $v_x^{(n)}$ (resp. $v_x^{(1)}$) as pointed out by the continuity relation (15).

C. Input and output interfaces

Using the same restrictions as mentioned at the end of Sec. I A, let us denote the (4×4) matrix $\mathscr{U}$:

$$\mathscr{U} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} \mathscr{S}^L \\ \mathscr{D}^L \\ \mathscr{D}^S \\ \mathscr{S}^S \end{bmatrix}_z = \mathscr{U} \begin{bmatrix} a_+^L \exp(jk_z^L z) \\ a_+^S \exp(jk_z^S z) \\ a_-^L \exp(-jk_z^L z) \\ a_-^S \exp(-jk_z^S z) \end{bmatrix}_z, \tag{17}$$

the inverse relation being derived from $\mathscr{U}^{-1}$.

When the input medium (0) is a solid, there is on Π_0:

$$\begin{bmatrix} v_x^{(0)} \\ v_z^{(1)} \\ \omega T_z^{(1)} \\ \omega T_x^{(1)} \end{bmatrix}_0 = \mathscr{M}^{(0)} \times \mathscr{U} \begin{bmatrix} a_+^L = a_i^L \\ a_+^S = a_i^S \\ a_-^L = a_r^L \\ a_-^S = a_r^S \end{bmatrix}_0 je^{j(Kx-\omega t)}, \tag{18}$$

in which $\mathscr{M} \times \mathscr{U}$ is detailed in the Appendix. As previously mentioned with Eq. (16), this relation (18) is still correct if the medium (1) is a liquid, here by using $v_x^{(0)}$ in place of $v_x^{(1)}$, this latter value being used if the medium (1) is a solid.

If the output medium $(n+1)$ is a solid, a relation similar to the inverse of Eq. (18) could be used on the last interface (z_n) with the matrix $\mathscr{U}^{-1} \times (\mathscr{M}^{(n+1)})^{-1}$. However, the matrix $\mathscr{M}^{-1}$ contains elements in which the inverses of k_z^L and k_z^S appear, which may induce numerical overflows when $|k_z^L|$ or $|k_z^S|$ are very small. But the definition of the output medium implies that $a_-^L(z) = 0$ and $a_-^S(z) = 0$, for any $z \geqslant z_n$. Defining

$$\Phi = a_+^L \exp(jk_z^L z_n) \quad \text{and} \quad \Psi = a_+^S \exp(jk_z^S z_n), \tag{19}$$

the difficulty can be overcome by noticing that in the medium $(n+1)$, there is (at z_n): $\mathscr{S}^L = \Phi$, $\mathscr{S}^S = \Psi$, $k_z^L(\mathscr{S}^L - \mathscr{D}^L) = 0$, and $k_z^S(\mathscr{S}^S - \mathscr{D}^S) = 0$. Thus the matrix $\mathscr{U}^{-1}$ can be replaced by the matrix $\mathscr{U}'$ which carries the transfer

$$\begin{bmatrix} \mathscr{S}^L \\ \mathscr{S}^S \\ k_z^L(\mathscr{S}^L - \mathscr{D}^L) \\ k_z^S(\mathscr{S}^S - \mathscr{D}^S) \end{bmatrix}_{z_n} = \mathscr{U}'^{(n+1)} \begin{bmatrix} \mathscr{S}^L \\ \mathscr{D}^L \\ \mathscr{D}^S \\ \mathscr{S}^S \end{bmatrix}_{z_n},$$

with

$$\mathscr{U}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ k_z^L & -k_z^L & 0 & 0 \\ 0 & 0 & k_z^S & -k_z^S \end{bmatrix}, \tag{20}$$

and thus enables use of the sparser product matrix $\mathscr{U}' \times \mathscr{M}^{-1}$ in place of $\mathscr{U}^{-1} \times \mathscr{M}^{-1}$, the former remaining correct even with null k_z^L or k_z^S:

($\mathscr{U}' \times \mathscr{M}^{-1}$ in the Appendix)

$$\begin{bmatrix} \Phi \\ \Psi \\ 0 \\ 0 \end{bmatrix}_{z_n} je^{j(Kx-\omega t)} = \mathscr{U}' \times (\mathscr{M}^{(n+1)})^{-1} \begin{bmatrix} v_x^{(n+1)} \\ v_z^{(n)} \\ \omega T_z^{(n)} \\ \omega T_x^{(n)} \end{bmatrix}_{z_n}, \tag{21}$$

with which the continuity relation $v_x^{(n)} = v_x^{(n+1)}$ can be used if the medium (n) is a solid.

D. Stack of solid layers between two liquid media

A single solid layer, or stack of solid layers, is supposed to be described with the (4×4) matrix $\mathscr{A}$ [Eq. (16)]. The following indexes are used: (0) and $(n+1)$ stand for the liquid media; (1) to (n) stand for the solid layer(s); z_0 and z_n are the abscissa of the mixed interfaces (0)–(1) and (n)–$(n+1)$. At these interfaces, v_x is not continuous and the T_x's are null:

$$\omega T_x^{(n+1)}(z_n) = \omega T_x^{(n)}(z_n) = \omega T_x^{(1)}(z_0) = \omega T_x^{(0)}(z_0) = 0, \tag{22}$$

from which the following can be derived at $z = z_0$:

$$\mathscr{A}_{41} v_x^{(1)} = -(\mathscr{A}_{42} v_z^{(0)} + \mathscr{A}_{43} \omega T_z^{(0)}). \tag{23}$$

The stack of solids, making a sandwich between two liquids, can thus be described by a (2×2) matrix, $\mathscr{B}'$, which acts in a manner similar to the transfer matrix used with liquid layers that will be given in Sec. II:

$$\begin{bmatrix} v_z^{(n+1)} \\ \omega T_z^{(n+1)} \end{bmatrix}_{z_n} = \mathscr{B}' \begin{bmatrix} v_z^{(0)} \\ \omega T_z^{(0)} \end{bmatrix}_{z_0}, \tag{24}$$

with

$$\mathscr{B}' = \begin{bmatrix} \mathscr{A}_{22} - \mathscr{A}_{21}\mathscr{A}_{42}/\mathscr{A}_{41} & \mathscr{A}_{23} - \mathscr{A}_{21}\mathscr{A}_{43}/\mathscr{A}_{41} \\ \mathscr{A}_{32} - \mathscr{A}_{31}\mathscr{A}_{42}/\mathscr{A}_{41} & \mathscr{A}_{33} - \mathscr{A}_{31}\mathscr{A}_{43}/\mathscr{A}_{41} \end{bmatrix}.$$

E. Notes about liquid–solid interfaces

Three comments about liquid–solid interfaces will be useful in Secs. III and IV and must be emphasized here. The geometry of the previous Sec. I D is used to introduce these notes.

(1) At the abscissa z_0, such a mixed interface implies a null component T_x. Thus, the fourth column of the matrix $\mathscr{A}$ [Eq. (16)] is not used to build $\mathscr{B}'$. So, the (4×3) submatrix made of the three first columns of the (4×4) matrix $a^{(1)}$ related to the first solid layer of the stack is sufficient to calculate the useful (4×3) submatrix of $\mathscr{A}$.

(2) On the other hand, the null value of T_x at the abscissa z_n of the second liquid–solid interface provides a condition (fourth row of $\mathscr{A}$) for the vector $[v_x^{(1)}(z_0), v_z^{(0)}(z_0), \omega T_z^{(0)}(z_0)]$ [Eq. (23)]. This allows removing the component $v_x^{(1)}(z_0)$ and thus leads to reducing the dimension of the final transfer matrix $\mathscr{B}'$ to (2×2). The pertinent fact that allows this reduction is the existence of a relation between the components finally transferred $[v_z^{(0)}(z_0), \omega T_z^{(0)}(z_0)]$ and a third component $[v_x^{(1)}(z_0)]$. Thus it is not essential that the first elementary matrix is strictly that one which is related to the transfer given by Eq. (14), i.e., $a^{(1)}$, but may be another matrix $a'^{(1)}$ related for example to such a transfer:

$$\begin{bmatrix} v_x^{(1)} \\ v_z^{(1)} \\ \omega T_z^{(1)} \\ \omega T_x^{(1)} \end{bmatrix}_{z_1} = \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \\ a'_{41} & a'_{42} & a'_{43} \end{bmatrix} \begin{bmatrix} a^u \\ v_z^{(0)} \\ \omega T_z^{(0)} \end{bmatrix}_{z_0}, \quad (\omega T_x^{(0)} = 0) \tag{25}$$

in which a^u denotes the amplitude of one of the waves that travel across the medium (1). We take advantage of this alternative to process the case of evanescent longitudinal waves in solid layers (Sec. III). The same reasoning still applies by using another relation between the initial components (here at z_0) than the one that induces the null value of T_x at z_n: this condition may be taken as the null value of the backward wave a_-^L in the output medium, as it happens in Eq. (21). This variation is used in Sec. V.

(3) It can be seen that the first row of $\mathscr{A}$ is not necessary to calculate $\mathscr{B}'$: the component v_x is not continuous at the abscissa z_n. Near such an interface, the matrix describing the solid part (here the last layer of the stack, i.e., $a^{(n)}$) can be the (3×4) submatrix composed of only the last three rows.

II. LIQUIDS

A. The transfer matrices

The relations to describe liquids are simpler than with solids in that the acoustic velocities $\mathbf{v}$ are derived from only a scalar potential ϕ, i.e., $\mathbf{v}=\text{grad}(\phi)$. The stress tensor σ is related to the strain rate tensor ϵ by the reduced expression $\sigma=\lambda\theta\mathbf{1}$ [see Eqs. (2) and (3)]. Thus, using the same conventions as in Sec. I A, it follows:

$$v_x=\frac{\partial\phi}{\partial x},\quad v_z=\frac{\partial\phi}{\partial z},\quad \sigma_{xz}=\sigma_{yz}=0,$$
$$\sigma_{zz}=T_z=j\frac{\lambda}{\omega}\left\{\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial x^2}\right\}. \tag{26}$$

It must also be emphasized that $T_x=0$. The fluid is viscous, so that the coefficient λ is a complex number. The complex square of the wave vector $\mathbf{k}^L$ is given by

$$(\mathbf{k}^L)^2=\rho\omega^2/\lambda. \tag{27}$$

For a given projection $\mathbf{K}$, the potential ϕ in a plane Π_z is written according to Eq. (7a). It leads to the matrix relation

$$\begin{bmatrix} v_z\\ \omega T_z\end{bmatrix}_z=\mathscr{N}\begin{bmatrix}\mathscr{S}^L\\ \mathscr{D}^L\end{bmatrix}_z je^{j(Kx-\omega t)}$$

with

$$\mathscr{N}=\begin{bmatrix}0 & k_z^L\\ -\rho\omega^2 & 0\end{bmatrix}. \tag{28}$$

The inverse relation is obtained with $\mathscr{N}^{-1}$.

Within a liquid layer, the (2×2) matrix $\ell(d)$ to transfer from a plane Π_z to a plane Π_{z+d} is finally written:

$$\begin{bmatrix} v_z\\ \omega T_z\end{bmatrix}_{z+d}=\ell(d)\begin{bmatrix} v_z\\ \omega T_z\end{bmatrix}_z,$$
$$\text{with } \boxed{\ell(d)=\mathscr{N}\times\mathscr{P}^L(d)\times\mathscr{N}^{-1}} \tag{29}$$

[the matrices $\mathscr{N}^{-1}$ and $\ell(d)$ are detailed in the Appendix].

B. Input and output interfaces

Let us denote $\mathscr{V}$ the (2×2) matrix so that at Π_z:

$$\begin{bmatrix}\mathscr{S}^L\\ \mathscr{D}^L\end{bmatrix}_z=\mathscr{V}\begin{bmatrix}a_+^L\ \exp(jk_z^Lz)\\ a_-^L\ \exp(-jk_z^Lz)\end{bmatrix}_z \text{ with } \mathscr{V}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}. \tag{30}$$

The inverse relation is obtained with $\mathscr{V}^{-1}$.

If the input medium (0) is a liquid, it gives

$$\begin{bmatrix} v_z^{(1)}\\ \omega T_z^{(1)}\end{bmatrix}_0=\mathscr{N}^{(0)}\times\mathscr{V}\begin{bmatrix}a_+^L=a_i^L\\ a_-^L=a_r^L\end{bmatrix}_0 je^{j(Kx-\omega t)},$$
$$\mathscr{N}\times\mathscr{V}=\begin{bmatrix}k_z^L & -k_z^L\\ -\rho\omega^2 & -\rho\omega^2\end{bmatrix}. \tag{31}$$

If the output medium $(n+1)$ is a liquid, there is by definition $a_-^L(z)=0\quad(z\geqslant z_n)$. Denoting $\Phi=a_+^L\ \exp(jk_z^Lz_n)$, one could write a relation which associates $(v_z^{(n)},\omega T_z^{(n)})$ with $(\Phi,0)$ at z_n, as the inverse relation of Eq. (31) by using $\mathscr{V}^{-1}\times(\mathscr{N}^{(n+1)})^{-1}$. But the same difficulty as mentioned in Sec. I C may occur when $|k_z^L|$ vanishes. The problem is still solved by noticing that into the medium $(n+1)$, at $z=z_n$, we have $\mathscr{S}^L=\Phi$ and $k_z^L(\mathscr{S}^L-\mathscr{D}^L)=0$. Thus, the matrix $\mathscr{V}'$ can be taken in place of $\mathscr{V}^{-1}$ to provide the following transfer:

$$\begin{bmatrix}\mathscr{S}^L\\ k_z^L(\mathscr{S}^L-\mathscr{D}^L)\end{bmatrix}_{z_n}=\mathscr{V}'^{(n+1)}\begin{bmatrix}\mathscr{S}^L\\ \mathscr{D}^L\end{bmatrix}_{z_n},$$

with

$$\mathscr{V}'=\begin{bmatrix}1 & 0\\ k_z^L & -k_z^L\end{bmatrix}, \tag{32}$$

so that it leads to the relation that is correct in all cases:

$$\begin{bmatrix}\Phi\\ 0\end{bmatrix}_{z_n} je^{j(Kx-\omega t)}=\mathscr{V}'\times(\mathscr{N}^{(n+1)})^{-1}\begin{bmatrix} v_z^{(n)}\\ \omega T_z^{(n)}\end{bmatrix}_{z_n} \tag{33}$$

($\mathscr{V}'\times\mathscr{N}^{-1}$ is given in the Appendix).

C. Stack of liquid layers and solids surrounded by liquids

Let us consider n contiguous liquid layers, each of them with the related matrix $\ell_i(d_i)$ (same sketch as Fig. 2). The media (0) and $(n+1)$ can be either liquid or solid. At the interfaces $(i+1)$–(i), the following relations of continuity are always true:

$$\omega T_x^{(i+1)}=\omega T_x^{(i)}=0\quad(\text{at } z_i)$$

and

$$\begin{bmatrix} v_z^{(i+1)}\\ \omega T_z^{(i+1)}\end{bmatrix}_{z_i}=\begin{bmatrix} v_z^{(i)}\\ \omega T_z^{(i)}\end{bmatrix}_{z_i}\quad(i=0,n). \tag{34}$$

In all cases, the stack can be characterized by a global (2×2) matrix $\mathscr{B}$ such that

$$\begin{bmatrix} v_z^{(n+1)}\\ \omega T_z^{(n+1)}\end{bmatrix}_{z_n}=\mathscr{B}\begin{bmatrix} v_z^{(0)}\\ \omega T_z^{(0)}\end{bmatrix}_{z_0},$$
$$\text{with } \boxed{\mathscr{B}=\ell^{(n)}(d_n)\times\cdots\times\ell^{(1)}(d_1)}, \tag{35}$$

with the additional conditions: $\omega T_x^{(n+1)}|_{z_n}=\omega T_x^{(0)}|_{z_0}=0$.

On the other hand, it has been seen in Sec. I D that a stack of solid layers surrounded by two liquid media is described with a global (2×2) matrix $\mathscr{B}'$ [Eq. (24)]. These solid stacks can then be naturally inserted as pseudoliquids in the liquid stacks, by associating these matrices $\mathscr{B}'$ with the $\ell^{(i)}$'s to build the global matrices $\mathscr{B}$ [Eq. (35)].

III. EVANESCENT LONGITUDINAL WAVES IN SOLID MEDIA

The so-called evanescent longitudinal waves occur when the vector **K** is so large that solving Eq. (6) leads to a value of $k_z^L = k_r^L + j\alpha^L$ that is mostly imaginary, i.e., $\alpha^L \gg k_r^L$. It must be emphasized that the basic process for solid layers described in Sec. I A still applies if the thickness of the solid layer, d, is small. The present method must only be undertaken if the condition $\exp(-\alpha^L d) \ll 1$ is met, i.e., the amplitude coefficient due to the absorption is vanishing. Hence, the transfer matrix $\mathscr{a}(d)$ becomes singular. The physical meaning of the singularity is the lack of causality between the mechanical state of the face $z+d$ and the evanescent wave a_+^L starting from the interface z (and conversely between the mechanical state at z and the backward wave a_-^L). There is no way to overcome this problem by increasing the precision of the computations. As far as there is no physical meaning in considering transmitted evanescent waves whose amplitudes are far beyond the range of precision of the machine, it can be said that the results obtained by using the method presented here are exact.

The solid layer, of which the bounding abscissa are denoted z and $z+d$, is described by a sequence of three matrices: a central (2×2) matrix which transfers the fictitious components v_z^l and T_z^l of a pseudo-material propagating only the shear waves actually traveling between interfaces z and $z+d$ (no longitudinal waves into this material: $a_+^L = a_-^L = 0$); and two matrices linking these fictitious mechanical states with the actual values. These matrices must be packaged (1) to remain compatible with the surrounding media, (2) so that the central matrix can be "seen" as that of a liquid.

A. The central matrix transferring the shear waves

The (2×2) matrix $\mathscr{b}'$ which associates the fictitious components $(v_z^l, \omega T_z^l)$ on both faces of the layer is derived first. These mechanical values are related only to the present shear waves, whereas the (evanescent) longitudinal waves starting from the interfaces are omitted. Within every plane Π_z into this pseudolayer, the (2×2) matrix $\mathscr{N}'$ is extracted from $\mathscr{M}$ [Eq. (12)] by taking into account $\mathscr{S}^L = \mathscr{D}^L = 0$:

$$\begin{bmatrix} v_z^l \\ \omega T_z^l \end{bmatrix}_{z'} = \mathscr{N}' \begin{bmatrix} \mathscr{D}^S \\ \mathscr{S}^S \end{bmatrix}_{z'} j e^{j(Kx-\omega t)},$$

with

$$\mathscr{N}' = \begin{bmatrix} 0 & K \\ -2\mu k_z^S K & 0 \end{bmatrix}. \tag{36}$$

Thus, using $\mathscr{N}'^{-1}$ and the propagation matrix $\mathscr{P}^S$ [Eq. (9)], one can build the transfer relation between Π_z and Π_{z+d}:

$$\begin{bmatrix} v_z^l \\ \omega T_z^l \end{bmatrix}_{z+d} = \mathscr{b}'(d) \begin{bmatrix} v_z^l \\ \omega T_z^l \end{bmatrix}_{z},$$

$$\text{with } \boxed{\mathscr{b}'(d) = \mathscr{N}' \times \mathscr{P}^S(d) \times \mathscr{N}'^{-1}} \tag{37}$$

($\mathscr{N}'^{-1}$ and $\mathscr{b}'$ are given in the Appendix).

B. Matrix related to the input of the layer

The interface located at the lowest abscissa, z, is referred here as the input side. By using the third and fourth rows of the matrix $\mathscr{M}^{-1}$, the actual mechanical components into this plane (solid layer side) are associated with $(\mathscr{D}^S, \mathscr{S}^S)$:

$$\begin{bmatrix} \mathscr{D}^S \\ \mathscr{S}^S \end{bmatrix}_z j e^{j(Kx-\omega t)} = \begin{bmatrix} -\dfrac{C_2}{k_z^S} & 0 & -\dfrac{K}{\rho\omega^2 k_z^S} & 0 \\ 0 & \dfrac{2S}{k^S} & 0 & \dfrac{1}{\rho\omega^2} \end{bmatrix} \begin{bmatrix} v_x \\ v_z \\ \omega T_z \\ \omega T_x \end{bmatrix}_z. \tag{38}$$

Then, two relations between the actual components and the fictitious values v_z^l, T_z^l are derived by linking Eqs. (36) and (38). A third relation between the actual mechanical components can be deduced by noticing that there is no backward longitudinal wave into this interface (by hypothesis, the wave a_-^L starting from Π_{z+d} is evanescent and cannot reach Π_z), so that $\mathscr{S}^L = \mathscr{D}^L$. Thus the last row of this input matrix is filled with the relation obtained by equating the two first rows in the inverse relation of Eq. (12):

$$\frac{2S}{k^S} v_x - \frac{C_2}{k_z^L} v_z - \frac{1}{\rho\omega^2}\omega T_z + \frac{K}{\rho\omega^2 k_z^L}\omega T_x = 0, \tag{39}$$

which is then formally compatible with a null component T_x in the pseudoliquid that describes $\mathscr{b}'$ in Sec. III A. Finally, the input interface matrix $\mathscr{a}_I'$ makes the transfer ($\mathscr{a}_I'$ in the Appendix)

$$\begin{bmatrix} v_z^l \\ \omega T_z^l \\ 0 \end{bmatrix}_z = \mathscr{a}_I' \begin{bmatrix} v_x \\ v_z \\ \omega T_z \\ \omega T_x \end{bmatrix}_z. \tag{40}$$

The size (3×4) of $\mathscr{a}_I'$ is sufficient to deal with all cases [note (3) in Sec. I E] because the solid pseudolayer (with a null thickness) that this matrix describes is followed by a layer formally processed as a liquid with the (2×2) matrix $\mathscr{b}'$. When the previous medium (abscissa lower than z) is a liquid, the matrix $\mathscr{a}_I'$ is reduced to the (2×2) matrix $\mathscr{B}_I'$ by using the method described in Section I D because the pseudolayer ($\mathscr{a}_I'$) is then formally surrounded by two liquids:

$$\begin{bmatrix} v_z^l \\ \omega T_z^l \end{bmatrix}_z = \mathscr{B}_I' \begin{bmatrix} v_z \\ \omega T_z \end{bmatrix}_z \quad \text{with } \mathscr{B}_I' = \begin{bmatrix} 2S^2 & 0 \\ \dfrac{\rho\omega^2 C_2^2}{k_z^L} & 1 \end{bmatrix}. \tag{41}$$

C. Output transfer matrix

On the other interface (i.e., the plane Π_{z+d}, solid layer side), the hypothesis is now $a_+^L \exp[jk_z^L(z+d)] = 0$, so that $\mathscr{S}^L = -\mathscr{D}^L$. This condition leads to the following deduction from the two first rows of $\mathscr{M}^{-1}$:

$$\frac{2S}{k^S} v_x + \frac{C_2}{k_z^L} v_z - \frac{1}{\rho\omega^2}\omega T_z - \frac{K}{\rho\omega^2 k_z^L}\omega T_x = 0. \tag{42}$$

On the other hand, according to the definition ($\mathscr{S}^L = \mathscr{D}^L = 0$) of the fictitious components v_x^l, v_z^l, ωT_z^l, and ωT_x^l, Eqs. (11) give

$$v_x^l = (k^S/2\rho\omega^2 S)\omega T_z^l \quad \text{and} \quad \omega T_x^l = (\rho\omega^2/K)C_2 v_z^l, \tag{43}$$

and also, by separating the contribution of the longitudinal wave,

$$\begin{bmatrix} v_x \\ v_z \\ \omega T_z \\ \omega T_x \end{bmatrix}_{z+d} = \begin{bmatrix} v_x^l \\ v_z^l \\ \omega T_z^l \\ \omega T_x^l \end{bmatrix}_{z+d} + \begin{bmatrix} K \\ -k_z^L \\ -\rho\omega^2 C_2 \\ \dfrac{2\rho\omega^2 S}{k^S} k_z^L \end{bmatrix} a_-^L e^{-jk_z^L(z+d)}. \tag{44}$$

Finally, assembling Eqs. (42)–(44), the output matrix a_0' of the layer is written (a_0' in the Appendix):

$$\begin{bmatrix} v_x \\ v_z \\ \omega T_z \\ \omega T_x \end{bmatrix}_{z+d} = a_0' \begin{bmatrix} a_-^L \exp[-jk_z^L(z+d)] \\ v_z^l \\ \omega T_z^l \end{bmatrix}_{z+d}. \tag{45}$$

As this pseudo solid layer is preceded by a fictitious medium which looks like a liquid with ℓ', the two first notes of Sec. I E apply and show that the reduced size, (4×3), of a_0' is sufficient, and justify the choice of the transferred components as well. In a similar way, as in Sec. III B, the method described in Sec. I D applies when the next layer (abscissa greater than $z+d$) is a liquid: The matrix a_0' is reduced to the following (2×2) matrix:

$$\begin{bmatrix} v_z \\ \omega T_z \end{bmatrix}_{z+d} = \mathscr{B}_0' \begin{bmatrix} v_z^l \\ \omega T_z^l \end{bmatrix}_{z+d} \quad \text{with } \mathscr{B}_0' = \begin{bmatrix} \dfrac{1}{2S^2} & 0 \\ \dfrac{\rho\omega^2 C_2^2}{2S^2 k_z^L} & 1 \end{bmatrix}. \tag{46}$$

IV. NORMAL ANGLE OF INCIDENCE ($K=0$)

When the direction of propagation is normal to the interfaces, the matrices $\mathscr{M}_a$ and $\mathscr{M}_b$ [Eqs. (11)] only have nonzero elements on the first and second diagonals respectively ($K=0, S=0, [C_2=1]$). The components v_z and T_z only depend on the longitudinal waves whereas v_x and T_x only depend on the shear waves. These shear waves are a special case of SH type waves, and are completely independent of the longitudinal waves. As the present paper is not concerned with this problem, the only elements of M_0 [Eq. (1)] that are calculated if $\mathbf{K}=\mathbf{0}$ are M_{11} and M_{21}, recalling that $M_{12}=M_{22}=M_{31}=M_{41}=0$. The two coefficients, M_{32} and M_{42}, are not computed in this case.

Thus the matrices describing the solid layers are here reduced to a (2×2) dimension, as are the ones related to liquids: If the input medium is a solid, a $\mathscr{N}\times\mathscr{V}$-like matrix [Eq. (31)] is still written, as is a $\mathscr{V}'\times\mathscr{N}^{-1}$ type matrix [Eq. (33)] at a solid output; all layers are also described with liquidlike matrix ℓ [Eq. (29)]. The ordered product (input–layers–output) of all these (2×2) matrices is a global (2×2) matrix, from which a partial inversion finally gives the desired coefficients M_{11} and M_{21} (see the next section).

V. LINKING

A. First steps

Summarized below is the sequence of the first operations to perform.

(1) Calculate the input matrix, $\mathscr{M}^{(0)}\times\mathscr{U}$ [Eq. (18)] if it is a solid, $\mathscr{N}^{(0)}\times\mathscr{V}$ [Eq. (31)] if it is a liquid (or a solid with $K=0$).

(2) Calculate the matrices related to each layer. The liquid (or solid with $K=0$) layers are characterized by $\ell^{(i)}$ matrices [Eq. (29)]. Each solid layer ($K\neq0$) is described by an $a^{(i)}$ matrix [Eq. (14)], unless it is a sequence of three matrices in the case of over attenuated longitudinal waves: $a_I'^{(i)}$ [Eq. (40)] or $\mathscr{B}_I'^{(i)}$ [Eq. (41)], $\ell'^{(i)}$ [Eq. (37)], and $a_0'^{(i)}$ [Eq. (45)] or $\mathscr{B}_0'^{(i)}$ [Eq. (46)], the choice of the matrices being used, a' or $\mathscr{B}'$, depending on the liquid or solid nature of the contiguous media.

(3) Calculate the output matrix: $\mathscr{U}'\times(\mathscr{M}^{(n+1)})^{-1}$ [Eq. (21)] for a solid ($K\neq0$); $\mathscr{V}'\times(\mathscr{N}^{(n+1)})^{-1}$ [Eq. (33)] when $K=0$ or the medium is a liquid.

(4) Calculate the products of all the similar contiguous matrices. It gives $\mathscr{A}$ matrices [Eq. (16)] with the solids, $\mathscr{B}$ matrices [Eq. (35)] with the liquids. It can be recalled that $\mathscr{M}\times\mathscr{U}$, a, a', and $\mathscr{U}'\times\mathscr{M}^{-1}$ matrices [Eqs. (18), (14), (40), (45), and (21)] are all compatible, and $\mathscr{N}\times\mathscr{V}$, ℓ, ℓ', $\mathscr{B}'$, and $\mathscr{V}'\times\mathscr{N}^{-1}$ types [Eqs. (31), (29), (37), and (33)] as well.

(5) Reducing of the liquid–solid–liquid sandwiches, providing $\mathscr{B}'$ type matrices [Eq. (24)]. If needed, a second pass performs the products of (2×2) liquidlike contiguous matrices that may remain after this step.

B. Final processing

No more than three matrices can be left after completing the steps of the previous Sec. V A. The simplest cases occur when there is only one matrix, $\mathscr{G}(2\times2)$ in case 1 or $\mathscr{H}(4\times4)$ in case 2. When two matrices remain, there is one of each type, $\mathscr{G}$ and $\mathscr{H}$. Two cases, depending on the input and output media may be encountered (cases 3 and 4). A last possibility may be found when the input and output media are both solid, and at least one layer is a liquid or equivalent (solid with evanescent longitudinal wave). A sequence of three matrices, $\mathscr{H}_a$, $\mathscr{B}$, and $\mathscr{H}_b$, must then be linked (case 5).

Case 1: Liquid input–[any layers]–liquid output, or $K=0$.

The partial resolution of

$$\begin{bmatrix}\Phi\\0\end{bmatrix}=\mathscr{G}\begin{bmatrix}a_i^L\\a_r^L\end{bmatrix}$$

gives

$$\boxed{\begin{bmatrix}\Phi\\a_r^L\end{bmatrix}=\begin{bmatrix}M_{11}\\M_{21}\end{bmatrix}[a_i^L].}$$

Within every other case (2–5), there is $K\neq 0$.

Case 2: Solid input–[solid layers (with transmitted longitudinal waves)]–Solid output.

The partial resolution of

$$\begin{bmatrix}\Phi\\\Psi\\0\\0\end{bmatrix}=\mathscr{H}\begin{bmatrix}a_i^L\\a_i^S\\a_r^L\\a_r^S\end{bmatrix}$$

gives

$$\boxed{\begin{bmatrix}\Phi\\a_r^L\\\Psi\\a_r^S\end{bmatrix}=M\begin{bmatrix}a_i^L\\a_i^S\end{bmatrix}.}$$

Case 3: Solid input–[any layers]–liquid output.

$$\begin{bmatrix}\Phi\\0\end{bmatrix}=\mathscr{G}\begin{bmatrix}v_z\\\omega T_z\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}v_z\\\omega T_z\\0\end{bmatrix}=\mathscr{H}'\begin{bmatrix}a_i^L\\a_i^S\\a_r^L\\a_r^S\end{bmatrix},$$

where $\mathscr{H}'$ is the (3×4) submatrix of $\mathscr{H}$ (see note 3 in Sec. I E).

Building the (3×3) submatrix $\mathscr{G}'$ from $\mathscr{G}$ so that

$$\mathscr{G}'=\begin{bmatrix} & & 0\\ & [\mathscr{G}] & 0\\ 0 & 0 & 1\end{bmatrix},$$

it gives

$$\begin{bmatrix}\Phi\\0\\0\end{bmatrix}=\mathscr{G}'\times\mathscr{H}'\begin{bmatrix}a_i^L\\a_i^S\\a_r^L\\a_r^S\end{bmatrix},$$

from which

$$\boxed{\begin{bmatrix}\Phi\\a_r^L\\a_r^S\end{bmatrix}=\begin{bmatrix}M_{11}&M_{12}\\M_{21}&M_{22}\\M_{41}&M_{42}\end{bmatrix}\begin{bmatrix}a_i^L\\a_i^S\end{bmatrix}}$$

can be derived.

Case 4: Liquid input–[any layers]–solid output.

$$\begin{bmatrix}\Phi\\\Psi\\0\\0\end{bmatrix}=\mathscr{H}'\begin{bmatrix}x\\v_z\\\omega T_z\end{bmatrix},$$

where $\mathscr{H}'$ is a (4×3) submatrix of $\mathscr{H}$ (see note 2 in Sec. I E) and

$$\begin{bmatrix}v_z\\\omega T_z\end{bmatrix}=\mathscr{G}\begin{bmatrix}a_i^L\\a_r^L\end{bmatrix}.$$

One of the two last rows within the first matrix equation enables the elimination of the component x, and thus building a (3×2) matrix $\mathscr{H}''$ which can be linked with $\mathscr{G}$:

$$\begin{bmatrix}\Phi\\\Psi\\0\end{bmatrix}=\mathscr{H}''\times\mathscr{G}\begin{bmatrix}a_i^L\\a_r^L\end{bmatrix}$$

which gives

$$\boxed{\begin{bmatrix}\Phi\\a_r^L\\\Psi\end{bmatrix}=\begin{bmatrix}M_{11}\\M_{21}\\M_{31}\end{bmatrix}[a_i^L].}$$

Case 5: Solid input–[any layers]–liquid layer (or solid layer with no transmitted evanescent longitudinal waves)–[any layers]–solid output.

The central block can be written in the form

$$\begin{bmatrix}v_z^b\\\omega T_z^b\end{bmatrix}=\mathscr{B}\begin{bmatrix}v_z^a\\\omega T_z^a\end{bmatrix},$$

whereas the input and output blocks are respectively written:

$$\begin{bmatrix}v_z^a\\\omega T_z^a\\0\end{bmatrix}=\mathscr{H}'_a\begin{bmatrix}a_i^L\\a_i^S\\a_r^L\\a_r^S\end{bmatrix},$$

where $\mathscr{H}'_a$ is a (3×4) submatrix of $\mathscr{H}$ (see note 3 in Sec. I E), and

$$\begin{bmatrix}\Phi\\\Psi\\0\\0\end{bmatrix}=\mathscr{H}'_b\begin{bmatrix}x\\v_z\\\omega T_z\end{bmatrix},$$

where $\mathscr{H}'_b$ is a (4×3) submatrix of $\mathscr{H}$ (see note 2 in Sec. I E).

As previously in case 4, the component x is eliminated to build the matrix $\mathscr{H}''_b$ which is combined with $\mathscr{B}$ and $\mathscr{H}'_a$ to make

$$\begin{bmatrix}\Phi\\\Psi\\0\\0\end{bmatrix}=\begin{bmatrix} & & 0\\ & [\mathscr{H}''_b\times\mathscr{B}] & 0\\ & & 0\\ 0 & 0 & 1\end{bmatrix}\times\mathscr{H}'_a\begin{bmatrix}a_i^L\\a_i^S\\a_r^L\\a_r^S\end{bmatrix},$$

the resolution of which gives

$$\boxed{\begin{bmatrix}\Phi\\ a_r^L\\ \Psi\\ a_r^S\end{bmatrix}=M\begin{bmatrix}a_i^L\\ a_i^S\end{bmatrix}}.$$

It may be noticed that the first four cases treat also the simplest combinations in which there is no stack of layer at all (single plane interfaces between two semi-infinite media).

There is a last numerical difficulty which may occur if the coefficients of transmission $(M_{11}, M_{12}, M_{31}, M_{32})$ are very small. In that case, the reflection coefficients must be calculated by limiting the number of layers taken into account: The only retained layers are the first m ones, counted from the input. The last (mth) layer is the first one to verify $\Pi_{i=1}^{m<n} \mathrm{Re}[\exp(jk_z^{u(i)} d_i)] \ll 1$ [u stands for L with liquids, S with solids; Re( ) means the real part of ( )].

The other layers ($i > m$) are not taken into account, and the mth layer acts as the output medium. The only significant coefficients are $(M_{21}, M_{22}, M_{41}, M_{42})$ whereas all the others are assigned a null value.

VI. APPLIED EXAMPLE

This example considers a longitudinal wave transmitted into a simulated kind of "araldite," at $\nu = 2.3$ MHz. We have been only interested with the evolution of the coefficient of transmission M_{11} versus the angle of incidence θ_0 $[K = (2\pi\nu/c_L^{(0)}) \sin\theta_0]$. The sequence of media is as follows (using the notations: $k^u = 2\pi\nu/c_u + j\alpha_u$, $u = L,S$):

solid input (0) (araldite): $\rho = 2000$ kg/m^3, $c_L = 1800$ m/s, $\alpha_L = 1000$ dB/m, $c_S = 1000$ m/s, $\alpha_S = 1000$ dB/m,

liquid layer (1): $\rho = 1140$ kg/m^3, $d = 0.05$ mm, $c_L = 1100$ m/s, $\alpha_L = 1000$ dB/m,

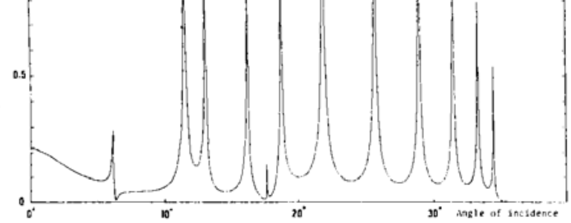

FIG. 3. Solid input–liquid layer (0.05 mm)–steel (6.3 mm)–water.

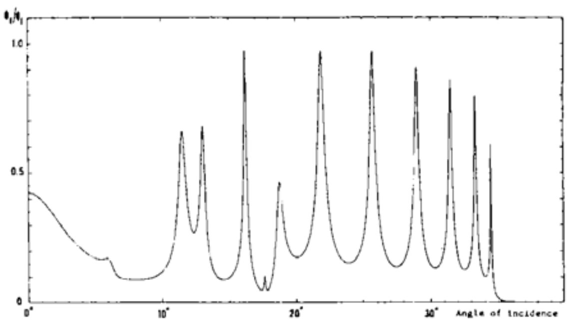

FIG. 4. Solid input–steel (6.3 mm)–water.

solid layer (2) (steel): $\rho = 7850$ kg/m^3, $d = 6.3$ mm, $c_L = 5900$ m/s, $\alpha_L = 10$ dB/m, $c_S = 3150$ m/s, $\alpha_S = 10$ dB/m,

liquid output (3) (water): $\rho = 1000$ kg/m^3, $c_L = 1450$ m/s, $\alpha_L = 1$ dB/m.

The result displayed in Fig. 3 is significantly different than in Fig. 4 for which the thin liquid layer (1) has been removed. The difference is due to the strong alteration of the tangential mechanical components due to the presence or absence of this layer. It can be noticed in Fig. 5 that a simulation including a liquidlike araldite input, without the liquid layer (1), gives a very similar result as the one in Fig. 3.

VII. CONCLUSION

The model presented here is suitable to deal with plane waves across plane layered media. The method of writing the expressions is developed keeping in mind the numerical problems involved during the actual processes, and these relations have been fully detailed. The easier case of the SH shear waves has not been taken up. A short example has been displayed, but much more complicated stacks of layers can

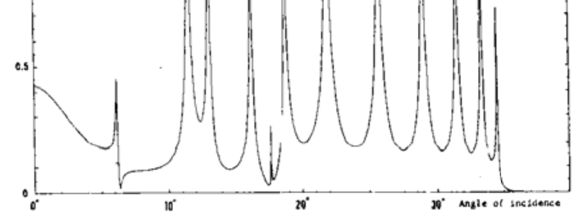

FIG. 5. Liquid input–steel (6.3 mm)–water.

be processed, and the transmission and reflection of complete fields can be undertaken as well, by using Fourier's formalism within the spatial frequency domain.

APPENDIX

The following notations are recalled: $S = K/k^S$, $C_2 = 1 - 2S^2$:

$$\mathscr{M} = \begin{bmatrix} K & 0 & -k_z^S & 0 \\ 0 & k_z^L & 0 & K \\ -\rho\omega^2 C_2 & 0 & -\dfrac{2\rho\omega^2 k_z^S S}{k^S} & 0 \\ 0 & -\dfrac{2\rho\omega^2 k_z^L S}{k^S} & 0 & \rho\omega^2 C_2 \end{bmatrix},$$

$$\mathscr{M}^{-1} = \begin{bmatrix} \dfrac{2S}{k^S} & 0 & -\dfrac{1}{\rho\omega^2} & 0 \\ 0 & \dfrac{C_2}{k_z^L} & 0 & -\dfrac{K}{\rho\omega^2 k_z^L} \\ -\dfrac{C_2}{k_z^S} & 0 & -\dfrac{K}{\rho\omega^2 k_z^S} & 0 \\ 0 & \dfrac{2S}{k^S} & 0 & \dfrac{1}{\rho\omega^2} \end{bmatrix}.$$

These additional notations are used to compress the next matrices:

$$\mathscr{m}^S = k_z^S S^S, \quad \mathscr{m}^L = k_z^L S^L,$$

$$\mathscr{d}^S = S^S/k_z^S, \quad \text{and} \quad \mathscr{d}^L = S^L/k_z^L.$$

It must be noticed that $\mathscr{d}^u(z) = j\sin(k_z^u z)/k_z^u$ takes the value jz with $k_z^u = 0$ (u in place of L or S). [S^L,C^L,C^S,S^S: See Eqs. (9).]

$$\mathscr{A} = \begin{bmatrix} C_2C^S + 2S^2C^L & KC_2\mathscr{d}^L - \dfrac{2S\mathscr{m}^S}{k^S} & K\dfrac{C^S - C^L}{\rho\omega^2} & -\dfrac{K^2\mathscr{d}^L + \mathscr{m}^S}{\rho\omega^2} \\ -\left(KC_2\mathscr{d}^S - \dfrac{2S\mathscr{m}^L}{k^S}\right) & C_2C^L + 2S^2C^S & -\dfrac{K^2\mathscr{d}^S + \mathscr{m}^L}{\rho\omega^2} & \mathscr{A}_{24} = \mathscr{A}_{13} \\ 2\rho\omega^2 SC_2\dfrac{C^S - C^L}{k^S} & -\rho\omega^2\left(C_2^2\mathscr{d}^L + \dfrac{4S^2\mathscr{m}^S}{(k^S)^2}\right) & \mathscr{A}_{33} = \mathscr{A}_{22} & \mathscr{A}_{34} = \mathscr{A}_{12} \\ -\rho\omega^2\left(C_2^2\mathscr{d}^S + \dfrac{4S^2\mathscr{m}^L}{(k^S)^2}\right) & \mathscr{A}_{42} = \mathscr{A}_{31} & \mathscr{A}_{43} = \mathscr{A}_{21} & \mathscr{A}_{44} = \mathscr{A}_{11} \end{bmatrix},$$

$$\mathscr{M} \times \mathscr{U} = \begin{bmatrix} K & -k_z^S & K & k_z^S \\ k_z^L & K & -k_z^L & K \\ -\rho\omega^2 C_2 & -\dfrac{2\rho\omega^2 k_z^S S}{k^S} & -\rho\omega^2 C_2 & \dfrac{2\rho\omega^2 k_z^S S}{k^S} \\ -\dfrac{2\rho\omega^2 k_z^L S}{k^S} & \rho\omega^2 C_2 & \dfrac{2\rho\omega^2 k_z^L S}{k^S} & \rho\omega^2 C_2 \end{bmatrix},$$

$$\mathscr{U}' \times \mathscr{M}^{-1} = \begin{bmatrix} \dfrac{2S}{k^S} & 0 & -\dfrac{1}{\rho\omega^2} & 0 \\ 0 & \dfrac{2S}{k^S} & 0 & \dfrac{1}{\rho\omega^2} \\ \dfrac{2Sk_z^L}{k^S} & -C_2 & -\dfrac{k_z^L}{\rho\omega^2} & \dfrac{K}{\rho\omega^2} \\ C_2 & \dfrac{2Sk_z^S}{k^S} & \dfrac{K}{\rho\omega^2} & \dfrac{k_z^S}{\rho\omega^2} \end{bmatrix},$$

$$\mathscr{N}^{-1} = \begin{bmatrix} 0 & -\dfrac{1}{\rho\omega^2} \\ \dfrac{1}{k_z^L} & 0 \end{bmatrix},$$

$$\mathscr{C} = \begin{bmatrix} C^L & -\dfrac{\mathscr{m}^L}{\rho\omega^2} \\ -\rho\omega^2\mathscr{d}^L & C^L \end{bmatrix},$$

$$\mathscr{V}' \times \mathscr{N}^{-1} = \begin{bmatrix} 0 & -\dfrac{1}{\rho\omega^2} \\ -1 & -\dfrac{k_z^L}{\rho\omega^2} \end{bmatrix},$$

$$\mathscr{N}'^{-1} = \begin{bmatrix} 0 & \dfrac{-k^S}{2\rho\omega^2 k_z^S S} \\ \dfrac{1}{K} & 0 \end{bmatrix},$$

$$\mathscr{C}' = \begin{bmatrix} C^S & -\dfrac{(k^S)^2\mathscr{d}^S}{2\rho\omega^2} \\ -\dfrac{2\rho\omega^2\mathscr{m}^S}{(k^S)^2} & C^S \end{bmatrix},$$

$$\mathscr{A}'_I = \begin{bmatrix} 0 & 2S^2 & 0 & \dfrac{K}{\rho\omega^2} \\ \dfrac{2\rho\omega^2 C_2 S}{k^S} & 0 & 2S^2 & 0 \\ \dfrac{2S}{k^S} & -\dfrac{C_2}{k_z^L} & -\dfrac{1}{\rho\omega^2} & \dfrac{K}{\rho\omega^2 k_z^L} \end{bmatrix},$$

$$
a_0' = \begin{bmatrix} K & 0 & \dfrac{k^S}{2\rho\omega^2 S} \\ -k_z^L & 1 & 0 \\ -\rho\omega^2 C_2 & 0 & 1 \\ \dfrac{2\rho\omega^2 S k_z^L}{k^S} & \dfrac{\rho\omega^2 C_2}{K} & 0 \end{bmatrix}.
$$

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