Fourier formalism for describing nonlinear self-demodulation of a primary narrow ultrasonic beam

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Presented here is the derivation of nonlinear interactions that occur within a primary narrow beam for which the temporal spectrum is continuous and narrow. This follows the bases of the Fourier formalism. Acoustics levels are presumed weak enough so that second-order equations may be used. In the quasilinear case, the exact theoretical expression of the created parametric farfield, formed from a transient modulated primary signal, is established, by using weakly restrictive assumptions. The case of high primary levels is discussed. Some experimental results are presented.

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INTRODUCTION

The idea of using the nonlinear properties of acoustical propagation arose about 1960 when Westervelt conceived the parametric emitter. In 1965, Berktay[1] suggested a way of producing pulsed signals by means of self-demodulating a primary wave in which the temporal spectrum does not remain discrete, but extends continuously through a narrow bandwidth. Using a plane-wave model, Berktay showed that a primary carrier, with the amplitude modulated by an envelope $f(t)$, produces a secondary signal in which the pattern is described with $\partial^2(f^2)/\partial t^2$. This phenomenon must not be confused with viscosity effects, studied within a linear framework.[2,3]

Since these theoretical expectations of Berktay, a certain amount of experimental confirmation has been achieved,[4–7] although another expression for the demodulated signal pattern ($\partial^2 f/\partial t^2$) was proposed by Merklinger[8] in the case of high primary level ($\Gamma \gg 1$; Γ is the Goldberg number). The inverse problem, i.e., the optimization of an input signal to obtain a given received pulse, has been studied by including numeral and experimental iterative processes.[9]

Some studies only deal with one-dimensional models, such as work[10] on the progressive distortion of an initially biharmonic plane wave in which the duration is limited to the difference-frequency period, with strong nonlinear interactions ($\Gamma \gg 1$). Other studies, based upon the Burger's equation, refer to plane, cylindrical, and spherical waves—i.e., solutions established for a few special cases[11]—and neglect linear absorption and those cases for which the validity is ensured only up to discontinuity length; or some studies refer to those exact solutions (plane waves)[12] derived from graphical methods that may be used to determine how wave profile is changing due to self-demodulation.

The complete three-dimensional (3-D) problem is treated in many ways. Using the Burger's equation again, Gurbatov *et al.*[13] examined the secondary directivity and the parametric efficiency of high-powered emitters ($\Gamma \gg 1$), for which the primary beams are supposed to be spherical from the source. Several calculations are supported by the Westervelt model; i.e., they use an integration of virtual sources in the volume where primary waves interact. Thus Moffett and Mello,[14] using temporal Fourier decomposition, obtained the theoretical expression of the secondary farfield, on-axis and far off-axis, with a source line hypothesis (collimated primary beam: $\alpha_0\mathscr{R}_0 \gg 1$; α_0 is the coefficient of linear absorption at the central primary frequency, and $\mathscr{R}_0$ is the Rayleigh distance of the transmitter at this frequency). With the hypothesis of spherical waves ($\alpha_0\mathscr{R}_0 \ll 1$), the result is given for all directions. Rolleigh[15] concluded that, for spherical primary sound fields in which the directivity follows a Gaussian law, the secondary sound field is given by the convolution of an input function $\{\partial^2(f^2)/\partial t^2\}$ with the impulse response of the parametric array. The theoretical approach of Pace and Ceen[16] is similar, but this impulse response is evaluated in the case of collimated plane-wave primary beams produced by a piston. More generally (Stephanishen and Koenigs[17]), this space-time impulse response may furthermore be seen itself as the convolution product of the impulse response of a linear array with another response that depends only upon the transducer aperture. For a circular piston, an original and simple geometrical approach[18] (numerical simulation and experiment) may also be mentioned: Formally, the on-axis response is evaluated by summing two waves. The first comes from the center of the disk, and the other from its contour, with the calculations taking into account the nonlinear distortion of these waves along their geometrical path. At the end, a low-pass filtering gives the demodulated signal.

Finally, mention may be made of a secondary field evaluation using parabolic approximation[19] and a discussion about the exact source location of the low-frequency signal observed in the farfield of a pulsed parametric emitter, i.e., to determine the effective interaction area from which the demodulated signal originates.[20]

Another interesting study (Gubatov and Dubkov[21]) involving nonlinear self-demodulation concerns the evaluation of the low-frequency noise generated by a quasimonochromatic source with small random phase and amplitude variations. In Ref. 22 is examined the stability in the response of a parametric emitter versus primary phase fluctuations.

Here is presented a theoretical calculus that gives the analytical expression of the secondary farfield generated by

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nonlinear self-demodulation of a primary 3-D wave. This primary field is radiated from a plane transducer, driven with a narrow frequency bandwidth. The source signal is thus obtained by modulating a carrier frequency ν_0. It is assumed that the primary directivity is such that the half-power beamwidth does not exceed about 10 deg (in fact, the only case with practical interest). A time-space Fourier analysis is used to describe this primary field which leads to a fairly general solution. Experimental results[7] test those theoretical results. The discrepancy between this theory and experiments when finite-amplitude waves occur is discussed. This paper extends and follows up previously presented studies.[23,24]

I. THE FOURIER FORMALISM IN ACOUSTICS (A REMINDER)

The notion of the angular spectrum of radiation has already been widely applied in acoustics.[25,26] But the Fourier analysis[27] technique cannot be directly generalized in absorbing media simply by using classical attenuated plane modes, in which the signature in the reference plane does not remain with a constant amplitude. This difficulty is overcome by using *inhomogeneous plane modes* introduced by Alais.[28,29] These modes are waves with equiphase planes that extend perpendicularly to the direction of propagation, while equiamplitude planes remain parallel with the reference plane. With these new elements of decomposition, the Fourier analysis is now applicable to problems of acoustics in absorbing media. Theoretical and numerical calculations induced by this method often exhibit simpler results through approximations that may be demonstrated. As far as our subject is concerned, the case of the nonlinear interaction of two such modes has been treated,[30] then extended to that of two harmonic radiations.[31] Many other developments followed later.[7,23,24,32–36] Presented here is the basis of the formalism involved in Sec. II.

A. Spatial Fourier decomposition

Let us consider a monochromatic radiation, propagating in an absorbing linear medium. Here, Π_0 ($z=0$) denotes a reference plane, so that there is no source in the half-space $z<0$. In every point $\mathbf{r}$ of this area, the acoustic field (pressure, voluminal mass, or velocity potential) $g(\mathbf{r},t) = G(\mathbf{r})\exp(-j\omega t)$ satisfies the classical wave equation

$$\Box' g = 0,$$

with (1)

$$\Box' \equiv \left(1 + \frac{\eta}{\rho_0 c_0^2}\frac{\partial}{\partial t}\right)\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2},$$

in which $\Box'$ denotes the operator of propagation in an absorbing medium (η is the global viscosity of the fluid).

We want to describe $G(\mathbf{r})$ as a superposition of plane waves that obey this equation. The Fourier technique is based upon a 2-D representation of fields, within Π_z planes which remain parallel to a reference plane Π_0, on which each elementary mode has a harmonic signature. But classical attenuated waves $G(\mathbf{r}) = \exp(j\,\mathbf{k}\cdot\mathbf{r} - \alpha r)$ do not satisfy this criterion.

The $G(\mathbf{q})$ distribution of the radiated field on Π_0 ($\mathbf{q}\in\Pi_0$) can be expressed by means of a classical decomposition:

$$G(\mathbf{q}) = \iint A_0^{(g)}(\mathbf{f})e^{2\pi j\mathbf{f}\cdot\mathbf{q}}\,d\mathbf{f}, \tag{2}$$

in which vectors $\mathbf{f}$ are the observed spatial frequencies in the reference plane Π_0. Conversely, the spectrum $A_0^{(g)}(\mathbf{f})$ is identified with the spatial Fourier transform of $G(\mathbf{r})$ in Π_0.

The formalism used requires that the plane modes suited to the decomposition have an amplitude spatial invariance along directions parallel to Π_0, except for the phase term $e^{2\pi j\mathbf{f}\cdot\mathbf{q}}$. Then, it is easy to show that the solutions sought take this form:

$$\exp[\,j(\mathbf{k}'\cdot r - \omega t]\,,$$

with

$$\mathbf{k}' = \mathbf{k} + j(\alpha/\cos\theta)\hat{\mathbf{z}}, \quad \mathbf{k} = 2\pi\mathbf{f} + \mathbf{k}_z, \tag{3}$$

and

$$|\mathbf{k}| = \frac{\omega}{c_0}, \quad \alpha = \frac{\omega^2\eta}{(2\rho_0 c_0^3)}, \quad \theta = (\mathbf{k},\mathbf{z}), \quad \sin\theta = \frac{2\pi f}{k},$$

taking into account $\alpha/k \ll 1$, and excluding far-axis modes (there is no $\cos\theta \ll 1$), which is not too restrictive an assumption.

So, these modes [Eq. (3)] possess a complex wave vector $\mathbf{k}'$ that obeys the dispersion law $k'^2(c_0^2 - j\omega\eta/\rho_0) = \omega^2$ associated with Eq. (1), and the projection of which within the spatial frequency plane remains real ($2\pi\,\mathbf{f}$). Consequently, these *inhomogeneous plane modes* remain compatible with the Fourier formalism since their equiamplitude planes remain parallel to the reference plane.

The complex amplitude $G(\mathbf{r})$ of a monochromatic acoustic radiation, at a point $\mathbf{r}$ of the plane Π_z (z is the distance between Π_z and the plane of emission), is then completely defined, within linear acoustics, by summing the inhomogeneous modes weighted by the associated spectral values, in the spatial frequency plane,

$$G(\mathbf{r}) = \iint A_0^{(g)}(\mathbf{f})\exp\left(-\frac{\alpha z}{\cos\theta} + j\mathbf{k}\cdot\mathbf{r}\right)d\mathbf{f}. \tag{4}$$

This relation may also be interpreted as the spatial inverse Fourier transform of the spectrum $A^{(g)}(\mathbf{f},z)$ related to the Π_z plane:

$$G(\mathbf{r}) = \iint A^{(g)}(\mathbf{f},z)e^{2\pi j\mathbf{f}\cdot\mathbf{m}}\,d\mathbf{f}, \quad \mathbf{m}\in\Pi_z \qquad (\mathbf{r} = \mathbf{m} + \mathbf{z}) \tag{5}$$

with $A^{(g)}(\mathbf{f},z) = A_0^{(g)}(\mathbf{f})H_{0z}(\mathbf{f})$. Here, $H_{0z}(\mathbf{f})$ is the operator of propagation from Π_0 to Π_z, related to the mode whose spatial frequency $\mathbf{f}$ gives the direction of propagation. Comparing Eqs. (4) and (5), it follows that

$$H_{0z}(\mathbf{f}) = \exp(-\alpha z/\cos\theta + jk_z z)\,. \tag{6}$$

When the distance z of the observation plane becomes greater ($z \gg \mathscr{R}_0$; $\mathscr{R}_0$ is the Rayleigh distance of the emitter), the integration of (4) with a saddle-point method leads to a farfield approximation, comparable to that of Fraunhofer:

$$G(\mathbf{r}) = (\cos\theta / j\lambda r) e^{2\pi j r/\lambda} e^{-\alpha r} A_0^{(g)}(\sin\theta/\lambda) . \quad (7)$$

[Here, θ is the observing angle $(\mathbf{r},\mathbf{z})$.]

When the emitter is plane, with pistonlike operation (front face uniformly moves along the z axis: $\mathbf{v} = v_z \hat{\mathbf{z}}$), the spectrum of acoustical velocities in Π_0, $A_0^{(v)}(\mathbf{f})$ reduces simply to the Fourier transform $\mathscr{A}(\mathbf{f})$ of the pupil function Pu describing the transducer aperture [$\mathrm{Pu}(\mathbf{q}) = 1$ on it, 0 elsewhere]:

$$A_0^{(v)}(\mathbf{f}) \equiv \mathscr{A}(\mathbf{f}) = \mathrm{FT}(\mathrm{Pu}(\mathbf{q})) = \iint e^{2\pi j \mathbf{f}\cdot\mathbf{q}} \, d\mathbf{q} . \quad (8)$$

Finally, let us note that relations $\mathbf{v} = \mathrm{grad}(\phi)$—i.e., $v_z = \partial\phi/\partial z$ in projection—and $\rho_0(\partial\mathbf{v}/\partial t) = -\mathrm{grad}(p) + \eta\nabla^2\mathbf{v}$—reduced to $p = -\rho_0(\partial\phi/\partial t)$ because of $\alpha/k \ll 1$—involve the following relations between potential, normal velocity, and pressure spectra of a monochromatic radiation:

$$A^{(v)} = jk_z A^{(\phi)}, \quad \text{and} \quad A^{(p)} = j\rho_0\omega A^{(\phi)}. \quad (9)$$

B. The Fourier formalism and nonlinear acoustics

The second-order equation of nonlinear propagation may be written

$$\Box'\phi = \mathrm{S}$$

with (10)

$$\mathrm{S} = \frac{1}{c_0^2}\frac{\partial}{\partial t}\left[(\nabla\phi)^2 + \frac{B}{2A}\frac{1}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2\right].$$

The existence of the source term S shows that space-time spectral components of radiated fields change during nonlinear propagation. In order to take this phenomenon into account, we still describe each harmonic beam (angular frequency ω) as a superposition of inhomogeneous plane modes [Eq. (3)], but with z-varying amplitudes, for every spatial frequncy $\mathbf{f}$:

$$g(\mathbf{r},t) = G(\mathbf{r})\exp(-j\omega t) ,$$

with

$$G(\mathbf{r}) = A_{0\omega}(\mathbf{f},z)\exp(-\alpha z/\cos\theta + j\mathbf{k}\cdot\mathbf{r}) . \quad (11)$$

With each considered angular frequency ω, this form allows the term of propagation $H_{0z}^{(\omega)}(\mathbf{f})$ [Eq. (6)], coming from the operator [Eq. (1)], to be kept in the expression of the spectrum $A_{z\omega}(\mathbf{f})$ corresponding to the Π_z plane:

$$A_{z\omega}(\mathbf{f}) = A_{0\omega}(\mathbf{f},z) H_{0z}^{(\omega)}(\mathbf{f},z) . \quad (12)$$

It should be noticed that evolutions of $A_{0\omega}(\mathbf{f},z)$, due to nonlinear propagation, remain slow compared with the z dependence of $H_{0z}^{(\omega)}$. Consequently, keeping the product [Eq. (12)] makes it easier to find solutions of Eq. (10). Solving it can now be reduced to calculating these reference spectra $A_{0\omega}(\mathbf{f},z)$, as their invariance (with respect to z)—in the linear model—is now destroyed.

These distributions $A_{0\omega}(\mathbf{f},z)$ are defined for every observation plane Π_z and every angular frequency ω. They may be understood as spectra returned to the $z = 0$ plane, which correspond to virtual sources, entirely localized in the half-space $z \leqslant 0$; and these sources produce the actually observed acoustic fields in Π_z. In every observation plane, it is then possible to calculate the acoustic field at a given angular frequency, transforming with Eq. (4) the corresponding spectrum of virtual emission, in the same way as with the linear model, including the ability to use the farfield approximation [Eq. (7)].

This model considers binary wave interactions as they appear with the quadratic nature of the source term S. Thus the elementary problem can be reduced to the interaction between two modes g_i [Eq. (11)], characterized by the two sets of angular and spatial frequencies $(\omega_i,\mathbf{f}_i,\ i = 1,2)$. The source term S [Eq. (10)] holds with elements at frequencies $2\omega_1$, $2\omega_2$, $\omega_+ = \omega_1 + \omega_2$, and $\omega_- = \omega_2 - \omega_1$ (choosing $\omega_2 > \omega_1$). We are only interested here in the coupling at the difference angular frequency ω_-. The corresponding source term S_- is thus written

$$\mathrm{S}_- = \frac{1}{c_0^2}\frac{\partial}{\partial t}\left(\nabla g_1^* \cdot \nabla g_2 + \frac{B}{2A}\frac{1}{c_0^2}\frac{\partial g_1^*}{\partial t}\frac{\partial g_2}{\partial t}\right). \quad (13)$$

The equation of propagation [Eq. (10)] here takes the form

$$\Box' g_- = \mathrm{S}_- , \quad (14)$$

from which one deduces, considering time invariance, that created g_- modes have the angular frequency ω_-:

$$g_-(\mathbf{r},t) = G_-(\mathbf{r})\exp(-j\omega_- t) . \quad (15)$$

From Eqs. (11) and (13), S_- is invariant in planes normal to $\mathbf{z}$, except for the phase argument $2\pi\mathbf{f}_-\cdot\mathbf{r}$, with $\mathbf{f}_- = \mathbf{f}_2 - \mathbf{f}_1$. Therefore, G_- modes share the same property and must be written

$$G_-(\mathbf{r}) = A_{0-}(z)\exp(-\alpha_- z/\cos\theta_- + j\mathbf{k}_-\cdot\mathbf{r}) , \quad (16)$$

with $\mathbf{k}_- + j(\alpha_-/\cos\theta_-)\hat{\mathbf{z}}$ the complex wave vector of the inhomogeneous mode associated with the frequencies $\mathbf{f}_-$ and ω_- [$\mathbf{k}_- = \omega_-\mathbf{n}_-/c_0$, $\sin\theta_- = 2\pi f_-/k_-$, and $\theta_- = (\mathbf{z},\mathbf{n}_-)$; $\mathbf{n}_-$ is a unit vector].

In Eq. (16), the exponential term is the solution of Eq. (1). Thus solving Eq. (14) can be reduced to a one-dimensional problem. Now, within the second-order approximation, the amplitude variations are weak after propagation along wavelengthlike distances; i.e., $|kA| \gg |\partial A/\partial z|$ and $|k(\partial A/\partial z)| \gg |\partial^2 A/\partial z^2|$. So, Eq. (14) finally results in a first-order differential equation. It can be noticed that vectors $\mathbf{k}_-$ and $(\mathbf{k}_2 - \mathbf{k}_1)$ have the same projection $(2\pi\mathbf{f})$ in the spatial frequency plane. Thus the phase difference $(\Delta\mathbf{k})\cdot\mathbf{r}$ between source and created wave is the following projection (on the $\mathbf{z}$ axis):

$$(\Delta\mathbf{k})\cdot\mathbf{r} = |\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{k}_-| z = (k_{z_2} - k_{z_1} - k_{z_-})\, z , \quad (17)$$

which allows one to write, in the paraxial case (directions $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_-$ are close to $\mathbf{z}$):

$$\frac{dA_{0-}(f_-,z)}{dz} = -\beta\frac{\omega_1\omega_2}{2c_0^3} A_{01}^*(\mathbf{f}_1,z) A_{02}(\mathbf{f}_2,z) e^{-v_- z} , \quad (18)$$

where $\beta = 1 + B/(2A)$, $v_- = \alpha - j\Delta k$, and $\alpha = \alpha_2 - \alpha_1 - \alpha_-$.

Within a plane Π_z, the contribution of both primary modes $(\mathbf{f}_1,\omega_1)$ and $(\mathbf{f}_2,\omega_2)$ to building $(\mathbf{f}_- = \mathbf{f}_2 - \mathbf{f}_1$, $\omega_- = \omega_2 - \omega_1)$ is given by integrating Eq. (18) along the z axis. When the pressure level of the studied acoustic fields remains low enough, the assumption of so-called weak interactions may be used to calculate the growth of modes created with nonlinearity. In this case, initial waves are supposed not to be altered; that is to say, primary waves obey the linear wave equation (1). Each primary amplitude $A_{0i}(\mathbf{f})$ no longer depends on z, and integration of Eq. (18) gives straightforwardly

$$A_{0-}(\mathbf{f}_-,z) = A^*_{01}(\mathbf{f}_1)A_{02}(\mathbf{f}_2)I_-(\mathbf{f}_1,\mathbf{f}_2,z), \tag{19}$$

where

$$I_- = -(\beta\omega_1\omega_2/2c_0^3)[(1-e^{-v_-z})/v_-].$$

In order to express v_-, it must be noted that the most constructive interaction between two primary modes is realized when the phase shift $\Delta\mathbf{k}$ [Eq. (17)] is null:

$$\Delta k(\mathbf{f}_{0_1},\mathbf{f}_{0_2}) = k_{z_2}(\mathbf{f}_{0_2}) - k_{z_1}(\mathbf{f}_{0_1}) - k_{z_-}(\mathbf{f}_-) = 0,$$

with

$$k_{z_i} = (\omega_i/c_0)\{1 - [(2\pi c/\omega_i)f_i]^2\}^{1/2}.$$

This occurs when the wave vectors $\mathbf{k}_{0_1}$ and $\mathbf{k}_{0_2}$ are colinear. Thus the corresponding spatial frequency vectors are described by

$$\frac{\mathbf{f}_{0_1}}{\omega_1} = \frac{\mathbf{f}_{0_2}}{\omega_2} = \frac{\mathbf{f}_-}{\omega_-}.$$

When $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_-$ points to close directions, it is then convenient to expand Δk with respect to this new spatial frequency vector:

$$\mathbf{f}' = (\omega_2\mathbf{f}_1 - \omega_1\mathbf{f}_2)/\omega_-,$$

as it can be seen that $\mathbf{f}_1 - \mathbf{f}_{0_1} = \mathbf{f}_2 - \mathbf{f}_{0_2} = \mathbf{f}'$. The expression of Δk_- developed to the second order then gives

$$\Delta k_- = \frac{2\pi^2 c_0\omega_-}{\omega_1\omega_2}\left[\mathbf{f}'^2 + \left(\frac{2\pi c_0\mathbf{f}_-\cdot\mathbf{f}'}{\omega_-\cos\theta_-}\right)^2\right],$$

which is reduced to $\Delta k_- = (2\pi^2 c_0\omega_-/\omega_1\omega_2)\mathbf{f}'^2$ in the paraxial case (as soon as $\tan\theta_- \ll 1$). Finally, the kernel I_- is reduced, in the farfield ($e^{-\alpha z}\ll 1$), to

$$I_{\infty -}(\mathbf{f}_1,\mathbf{f}_2) = -\frac{\beta\omega_1\omega_2}{2c_0^3\alpha}\left[1 - j\frac{2\pi^2 c_0\omega_1\omega_2}{\alpha\omega_1}\times\left(\frac{\mathbf{f}_1}{\omega_1} - \frac{\mathbf{f}_2}{\omega_2}\right)^2\right]^{-1}. \tag{20}$$

II. NONLINEAR SELF-DEMODULATION

A. Primary field

One assumes that a pistonlike transducer is fed with a harmonic (frequency ν_0) signal. This signal is modulated by a function v_0. On the plane Π_0 ($z=0$) containing this emitter, the normal velocity field may be written

$$v_z(\mathbf{q},t) = \mathrm{Pu}(\mathbf{q})v_0(t)\exp(-2\pi j\nu_0 t) \quad (\mathbf{q}\in\Pi_0), \tag{21}$$

in which Pu is the aperture function of the transducer.

Separation of variables z and t in the description of motion given in Eq. (21) allows the corresponding space-time spectrum to be obtained easily:

$$A^{(\omega)} = \mathscr{A}(\mathbf{f})\Upsilon(\nu-\nu_0), \tag{22}$$

in which $\Upsilon(\nu)$ is the temporal Fourier transform of the modulating function $v_0(t)$, and $\mathscr{A}(\mathbf{f})$ the spatial Fourier transform (8) of the pupil function $\mathrm{Pu}(\mathbf{q})$.

Let us note the typical mean primary high-frequency (HF) directivity angle of the transmitter as $\theta_d \simeq (\lambda_0/\mathscr{R}_0)^{1/2}$, in which $\mathscr{R}_0$ is the Rayleigh distance calculated at the central frequency (the shape of the aperture is supposed not to be too asymmetric). We assume θ_d is small enough compared with 1 so that the only significant spectral components $\mathscr{A}(\mathbf{f})$ [compared with $\mathscr{A}(\mathbf{0})$] are those that remain near the axis $\mathbf{z}$. Therefore, the approximations $\sin\theta\simeq\theta$ and $\cos\theta\simeq 1$ are valid in amplitude calculations, within the useful domain of spatial frequencies $\mathbf{f} = \nu\boldsymbol{\theta}/c_0$.

Following Eq. (9), the potential spectrum of the primary beam is then defined by

$$A_0(\mathbf{f},\nu) = (1/jk_z)A^{(\omega)} \simeq (c_0/j2\pi\nu)\mathscr{A}(\mathbf{f})\Upsilon(\nu-\nu_0). \tag{23}$$

With the assumption of weak nonlinear interactions, the primary radiated field obeys the wave equation (1). In terms of the potential, this field is written, given Eqs. (4) and (23), as

$$\Phi(\mathbf{r},t) = \int\frac{c_0}{j2\pi\nu}\Upsilon(\nu-\nu_0)\left(\iint\mathscr{A}(\mathbf{f})e^{j\mathbf{k}\cdot\mathbf{r}}\,d\mathbf{f}\right)\times\exp[-\alpha(\nu)r - 2\pi j\nu t]\,d\nu. \tag{24}$$

On the other hand, we assume the relative bandwidth to be narrow: The $\Upsilon(\nu)$ temporal spectrum of $v_0(t)$ extends only within low frequencies ν compared with ν_0. Noting $\nu_{\mathrm{BF\,max}}$ as the greatest extension of the bandwidth, we define a parameter $\hbar_m$ with

$$\hbar_m = \nu_0/\nu_{\mathrm{BF\,max}} \gg 1. \tag{25}$$

B. The spectrum of nonlinear interaction products

Self-demodulation results from parametric interactions between every space-time plane mode corresponding to the $\Upsilon(\nu)$ spectrum. The low-frequency acoustic field thus created is generally not so easy to evaluate because it depends upon the modulation function $v_0(t)$, through a space-time filter. However, this difficulty is removed when using the z–t separation to define the normal velocity [Eq. (21)], which is used for temporal and spatial frequencies with respect to the spectrum expression [Eq. (23)].

The interaction of two constitutent modes $[(\mathbf{f}_1,\nu_1),(\mathbf{f}_2,\nu_2)]$ of the primary field [Eq. (24)] creates the mode ($\mathbf{f} = \mathbf{f}_2 - \mathbf{f}_1$, $\nu = \nu_2 - \nu_1$). Far from the source area ($[\alpha(\nu_1) + \alpha(\nu_2) - \alpha(\nu)]z \gg 1$), and in the paraxial case ($f_1 \ll \nu_1/c_0$, $f_2 \ll \nu_2/c_0$, and $f \ll \nu/c_0$), this secondary mode is given with Eqs. (19) and (20), which become here

$$a_{0-}(\mathbf{f},\nu) = -(\beta/2c_0 v_-)\mathscr{A}^*(\mathbf{f}_1)\Upsilon^*(\nu_1-\nu_0)\times\mathscr{A}(\mathbf{f}_1+\mathbf{f})\Upsilon(\nu_1+\nu-\nu_0),$$

with

$$v_- = [\alpha(\nu_1) + \alpha(\nu_2) - \alpha(\nu)] - j[\pi c_0\nu/\nu_1(\nu_1+\nu)][\mathbf{f}_1 - (\nu_1/\nu)\mathbf{f}]^2. \tag{26}$$

The relative narrowness of the primary bandwidth [Eq. (25)] allows v_- to be reduced, using approximate expressions which no longer depend upon ν_1:

$\alpha(\nu_1) \simeq \alpha(\nu_2) \simeq \alpha(\nu_0) = \alpha_0, \quad \nu/[\nu_1(\nu_1+\nu)] \simeq \nu/\nu_0^2,$
and $\nu_1/\nu \simeq \nu_0/\nu$.

Thus (27)

$$\upsilon_- \cong 2\alpha_0\{1 - j(\pi c_0 \nu/2\nu_0^2\alpha_0)[\mathbf{f}_1 - (\nu_0/\nu)\mathbf{f}]^2\}.$$

Integrating Eq. (26) with respect to ν_1 and $\mathbf{f}_1$, so as to add every contribution that creates the secondary mode $(\mathbf{f},\nu)$, it appears, with the help of approximations (27), that a partial separation between temporal and spatial frequencies remains in the secondary spectrum expression:

$$A_{0-}(\mathbf{f},\nu) = -(\beta/4c_0\alpha_0)G(\nu)S(\nu,\theta = c_0\mathbf{f}/\nu), \quad (28)$$

with

$$G(\nu) = \int \Upsilon^*(\nu_1 - \nu_0)\Upsilon(\nu_1 + \nu - \nu_0)\, d\nu_1, \quad (29)$$

and

$$S(\nu,\theta) = \iint \mathscr{A}^*(\mathbf{f}_1)\mathscr{A}\left(\mathbf{f}_1 + \frac{\theta\nu}{c_0}\right) \times\left[1 - j\frac{\pi c_0 \nu}{2\nu_0^2\alpha_0}\left(\mathbf{f}_1 - \frac{\theta\nu_0}{c_0}\right)^2\right]^{-1} d\mathbf{f}_1. \quad (30)$$

C. Demodulated field

1. General expression

In the Frauhofer zone of the secondary radiation—that is to say, in a farfield of the primary interaction zone—the radiated field associated with the spectrum [Eq. (28)] is obtained with Eq. (7), for a given ν frequency and θ paraxial direction. Following Eq. (9), it is written in terms of pressure:

$$P_-(z,\theta,\nu) \simeq (2\pi j\nu)^2 G(\nu) e^{2\pi j\nu r/c_0} \times(\rho_0\beta/8\pi c_0^2\alpha_0 r)S(\nu,\theta)e^{-\alpha(\nu)r}. \quad (31)$$

Here, $G(\nu)$ [Eq. (29)] is the self-convolution product of the $\Upsilon(\nu)$ spectrum. From this property, it may be derived that the temporal inverse Fourier transform of the first three terms within the second part of Eq. (31) is

$$\mathrm{FT}_t^{-1}\left[(2\pi j\nu)^2 G(\nu)\exp\left(\frac{2\pi j\nu r}{c_0}\right)\right] = \frac{d^2}{dt^2}\left[\left|\upsilon_0\left(t - \frac{r}{c_0}\right)\right|^2\right]. \quad (32)$$

If $\upsilon_0(t)$ is a real function, which represents an exclusive amplitude modulation, $|\upsilon_0(t)|^2$ is the square of this envelope υ_0; otherwise it means the square of the modulus $|\upsilon_0|$ if the HF carrier is also phase modulated.

In other respects, one notices that $S(\nu,\theta)$ [Eq. (30)] is the farfield paraxial secondary directivity of the (directional) transducer working as a parametric transmitter with a $\hbar = \nu_0/\nu$ frequency ratio, calculated with the weak nonlinear interaction assumption. Let us denote $\Sigma(t,\theta)$ as the temporal inverse Fourier transform of S.

Finally, with the sole hypothesis that a transducer, which is supplied with a narrow-bandwidth modulated carrier, radiates a sufficiently directional HF beam, the demodulated paraxial farfield may be written

$$P_-(z,\theta,t) = \frac{\rho_0\beta}{8\pi c_0^2\alpha_0 r}\frac{d^2}{dt^2}\left[\left|\upsilon_0\left(t - \frac{r}{c_0}\right)\right|^2\right] * \Sigma(t,\theta) * \Lambda(r,t), \quad (33)$$

in which $\Lambda(r,t)$ is the temporal inverse Fourier transform of the $\exp[-\alpha(\nu)r]$ function, i.e., the impulse response of the absorbing medium after a unidimensional propagation along a distance r.

Usually, $\Sigma(t,\theta)$ can be obtained only by numerical calculus. However, with a few added assumptions, other analytical results can be obtained.

2. Approximate expression

The following developments are based upon the assumption that the HF beam directivity θ_d verifies $\theta_d^2 \ll c_0\alpha_0/\nu_{\mathrm{BF\,max}}$, this condition being equivalent to

$$\hbar_m \alpha_0 \mathscr{R}_0 \gg 1 \quad (34)$$

[$\hbar_m$: see Eq. (25); α_0: see Eq. (27); $\mathscr{R}_0$: Rayleigh distance (at ν_0)].

Since $\hbar_m \gg 1$ [Eq. (25)], it must be noticed that this assumption [Eq. (34)] does not mean that the zone where interactions take place is necessarily confined to the collimated part of the primary beam, as it is with the Westervelt model in which $\alpha_0\mathscr{R}_0 > 1$. It only states that the interaction zone does not extend beyond a distance of about $\hbar_m\mathscr{R}_0$.

On-axis $(\theta = 0)$, it can be seen that the product $\mathscr{A}^*(\mathbf{f}_1)\mathscr{A}(\mathbf{f}_1)$ involved in Eq. (30) takes its significant values when the spatial frequency $\mathbf{f}_1$ belongs to a domain in which the size does not extend beyond a distance $\theta_d\nu_0/c_0$ from the origin. Thus the following term may be ignored in the interaction kernel:

$$\pi c_0 \nu/(2\nu_0^2\alpha_0)\, \mathbf{f}_1^2 \ll 1, \quad (35)$$

and $S(\nu,0)$ is then reduced to the value of the transducer area $\mathscr{S}$:

$$\mathscr{S} = \iint \mathscr{A}^*(\mathbf{f}_1)\mathscr{A}(\mathbf{f}_1)\, d\mathbf{f}_1. \quad (36)$$

The demodulated farfield signal received on the axis then has the shape given by Berktay,[1] except the convolution with the Λ function (due to linear absorption):

$$P_-(z,0,t) = \frac{\rho_0\beta\mathscr{S}}{8\pi c_0^2\alpha_0 z}\frac{d^2}{dt^2}\left[\left|\upsilon_0\left(t - \frac{z}{c_0}\right)\right|^2\right] * \Lambda(z,t). \quad (37)$$

Off-axis, the observation angle θ, previously assumed to be paraxial, is now also restricted so that $\theta\nu/c_0 \ll \theta_d\nu_0/c_0$:

$$\theta \ll 1, \quad \theta \ll \hbar_m\theta_d. \quad (38)$$

This condition ensures that the product $\mathscr{A}^*(\mathbf{f}_1)\mathscr{A}(\mathbf{f}_1 + \theta\nu/c_0)$ in Eq. (30) takes its significant values when the spatial frequency $\mathbf{f}_1$ belongs to a doman in which the location does not extend beyond the on-axis case. Thus it still is possible to eliminate (35) in order to integrate Eq. (30).

To eliminate the cross term $(\pi\nu/2\nu_0\alpha_0)\mathbf{f}_1\theta$ as well, the observation angles θ are furthermore supposed to conform with

$$\theta \ll (\hbar_m\alpha_0\mathscr{R}_0)\theta_d, \quad (39)$$

which, taking into account Eq. (34), is not a more restrictive condition that the paraxial hypothesis already in effect.

The expression (30) of $S(\nu,\boldsymbol{\theta})$ is then reduced to

$$S(\nu,\boldsymbol{\theta}) = \left(1 - j\frac{\pi\boldsymbol{\theta}^2\nu}{2c_0\alpha_0}\right)^{-1} \times \iint \mathscr{A}^*(\mathbf{f}_1)\mathscr{A}\left(\mathbf{f}_1 + \frac{\boldsymbol{\theta}\nu}{c_0}\right) d\mathbf{f}_1 . \qquad (40)$$

The $\Sigma(t,\boldsymbol{\theta})$ Fourier transform of $S(\nu,\boldsymbol{\theta})$ thus appears to be the temporal convolution product: $\Sigma(t,\boldsymbol{\theta}) = \Theta(t,\theta) * \Xi(t,\boldsymbol{\theta})$, in which $\Theta(t,\theta)$ is the impulse response of the parametric source line and $\Xi(t,\boldsymbol{\theta})$ that of the transducer.

The first response is the temporal inverse Fourier transform of the Westervelt directivity function:

$$\Theta(t,\theta) = \int_{-\infty}^{+\infty} \left(1 - j\frac{\pi\nu\theta^2}{2c_0\alpha_0}\right)^{-1} e^{-2j\pi\nu t}\, d\nu , \qquad (41)$$

the expression of which can be shown to be

$$\Theta(t,\theta) = H(t)(4\alpha_0 c_0/\theta^2)\exp[-(4\alpha_0 c_0/\theta^2)t] , \qquad (42)$$

in which H is the Heavyside function $[H(t>0) = 1$, $H(0) = \frac{1}{2}$, and $H(t<0) = 0]$.

The calculation of $\Xi(t,\boldsymbol{\theta})$ is made by rotating the reference frame (f_x, f_y) of the spatial frequency plane so that one of the axes (for instance, f_y) equates the projection, on this plane, of the direction $\boldsymbol{\theta}$:

$$\Xi(t,\boldsymbol{\theta}) = \int_{-\infty}^{+\infty} \left[\iint \mathscr{A}^*(\mathbf{f}_1)\mathscr{A}\left(\mathbf{f}_1 + \frac{\boldsymbol{\theta}\nu}{c_0}\right) d\mathbf{f}_1\right] \times e^{-2j\pi\nu t}\, d\nu$$
$$= \frac{c_0}{\theta}\int\left(\int \mathscr{A}^*(f_x, f_y) e^{2\pi j c_0 f_y t/\theta}\, df_y\right) \times \left(\int \mathscr{A}(f_x, f'_y) e^{-2\pi j c_0 f'_y t/\theta}\, df'_y\right) df_x . \qquad (43)$$

Next, one replaces $\mathscr{A}$ with its definition, that is to say, the Fourier transform of the pupil function Pu over the transducer aperture (notice that $\mathrm{Pu}^* = \mathrm{Pu}$):

$$\Xi(t,\boldsymbol{\theta}) = \frac{c_0}{\theta}\int\left[\int\left(\int \mathrm{Pu}(x,y) e^{2\pi j(f_x x + f_y y)}\, dx\, dy\right) \times e^{2\pi j c_0 f_y t/\theta}\, df_y\right] \times \left[\int\left(\int \mathrm{Pu}(x',y') e^{-2\pi j(f_x x' + f'_y y')}\, dx'\, dy'\right) \times e^{-2\pi j c_0 f'_y t/\theta}\, df'_y\right] df_x . \qquad (44)$$

After resequencing the different integration variables, and computing parts that are relative to exponential terms, one obtains

$$\Xi(t,\boldsymbol{\theta}) = \frac{c_0}{\theta}\iiiint \mathrm{Pu}(x,y)\mathrm{Pu}(x',y') \times \delta(x - x')\delta(y + c_0 t/\theta) \times \delta(y' + c_0 t/\theta)\, dx\, dy\, dx'\, dy', \qquad (45)$$

which, taking into account that $\mathrm{Pu}^2(x,y) = \mathrm{Pu}(x,y)$, finally leads to

$$\Xi(t,\boldsymbol{\theta}) = \frac{c_0}{\theta}\int \mathrm{Pu}\left(x, -\frac{c_0 t}{\theta}\right) dx. \qquad (46)$$

The geometric interpretation of this last function is evident: $\Xi(t,\boldsymbol{\theta})$'s temporal evolution is proportional to the length $L_x(-c_0 t/\theta)$ of the segment where the transducer aperture intersects the plane that approximates the sphere centered at the observation point with a radius growing at the speed c_0:

$$\Xi(t,\boldsymbol{\theta}) = (c_0/\theta)L_x(-c_0 t/\theta). \qquad (47)$$

Thus $\Xi(t,\boldsymbol{\theta})$ is the classical farfield impulse response of the transducer.

To sum up, using only the following hypotheses: $\theta_d \ll 1$: high HF primary directivity; $\hbar_m \gg 1$: high parametric frequency ratio; $\hbar_m \alpha_0 \mathscr{R}_0 \gg 1$: interaction area does exceed the distance $\hbar_m \mathscr{R}_0$ beyond the transducer; $\theta \ll 1$, $\theta \leqslant \hbar_m \theta_d$, $\theta \ll (\hbar_m \alpha_0 \mathscr{R}_0)\theta_d$: paraxial direction of observation; the acoustic farfield [Eq. (33)] created by nonlinear self-demodulation of a primary HF beam is described, in terms of pressure, with the following sequence of convolution products:

$$P_-(z,\boldsymbol{\theta},t) = \frac{\rho_0 \beta \mathscr{S}}{8\pi c_0^2 \alpha_0 z}\frac{d^2}{dt^2}\left[\left|v_0\left(t - \frac{r}{c_0}\right)\right|^2\right] * \Theta(t,\theta) * \Xi(t,\boldsymbol{\theta}) * \Lambda(r,t). \qquad (48)$$

The first term (32) is closely dependent upon the modulating function; the second term (42) takes into account the structure of the interaction area; the third term (47) is dictated by the shape of the transducer; the last term reflects the viscous attenuation.

One can verify the consistent results $\int_{-\infty}^{+\infty} \Theta(t,\theta)\, dt = 1$ and $\int_{-\infty}^{+\infty} \Xi(t,\boldsymbol{\theta})\, dt = \mathscr{S}$. Furthermore, there are also $\Theta(t,0) = \delta(t)$ and $\Xi(t,\mathbf{0}) = \mathscr{S}\delta(t)$, which correctly return the on-axis result [Eq. (37)] from Eq. (48). On the other hand, characteristic temporal widths of functions Ξ and Θ are, respectively, given by $\Delta t_l = \theta^2/(4\alpha_0 c_0)$ (subscript l used to recall source line) and $\Delta t_s = \theta\ell/c_0$ (subscript s for transducer section; ℓ is its mean transverse size: $\ell^2 \simeq \mathscr{S}$). The effects of convolutions with these functins are even more important as the corresponding intervals $\Delta t(\theta)$ are not negligible compared with the minimal period $\Delta T = 1/\nu_{\mathrm{BFmax}}$. It must be noticed that the hypotheses that connect relative values of θ, θ_d, $\hbar_m$, and α_0 cannot *a priori* locate these intervals Δt_l and Δt_s with regard to ΔT. It is only possible to deduce the following relations: $\Delta t_l \ll (4\alpha_0\lambda)^{-1}\Delta T$, $\Delta t_s \ll (\hbar_m \theta_d)^{-1}\Delta T$; $\Delta t_l \leqslant \hbar_m (4\alpha_0 \mathscr{R}_0)^{-1}\Delta T$, $\Delta t_s \leqslant \Delta T$; $\Delta t_l \ll (\hbar_m \alpha_0 \mathscr{R}_0/4)\Delta T$, $\Delta t_s \ll (\alpha_0 \mathscr{R}_0)\Delta T$; in which nothing can be said about $\alpha_0\lambda$, $\hbar_m \theta_d$, and $\alpha_0 \mathscr{R}_0$ with regard to 1. However, an important fact to be considered is that the condition $\alpha_0 \mathscr{R}_0 \gg 1$ rarely occurs. So the last relation $\Delta t_s \ll \alpha_0 \mathscr{R}_0 \Delta T$ shows that, according to the specified hypotheses, the convolution with Ξ in Eq. (48) disappears and is replaced only with a product with the factor $\mathscr{S}$. Thus the exact shape of the transducer aperture has no effect upon the demodulated signal pattern within the paraxial farfield.

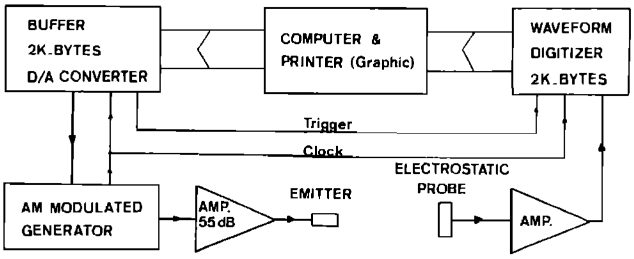

FIG. 1. Schematic of experiment.

3. High acoustic levels

When the emitted pressure level increases, it seems very difficult to develop a practical sophisticated model to describe nonlinear self-demodulation including finite-amplitude effects. The primary wave, in which the spectral domain is a narrow bandwidth centered in ν_0, dissipates energy by creating harmonic bands centered around frequencies $n\nu_0$ (n integer). But all these new components again interact among themselves, so that a simple analytical calculus is no longer available.

The low-frequency parametric radiation is no longer produced by the only interactions of the primary waves. The signal previously obtained from weak interactions should therefore be significantly distorted. A simplified argument in order to understand this phenomenon is that the self-demodulation of every harmonic band produces a wave that is superimposed upon the signal [Eq. (22)].

III. EXPERIMENTS (REF. 7)

We have begun an experimental investigation (in water) of the theoretical approach using the configuration described in Fig. 1. By means of the computer and buffer memory, any kind of modulation function can be used, and through the waveform digitizer, several received signals can be added together to raise the signal-to-noise ratio. The electrostatic probe has a flat response for low frequencies up to 1 MHz, and the emitting transducer centered at 6.2 MHz has a 3-dB bandwidth larger than 1 MHz. It behaves just like a flat oscillating 15-mm-diam piston with a high directivity: $\theta_{3\ \mathrm{dB}} = 0.6°$.

The characteristic values associated with the primary beam are $\mathscr{R}_0 \simeq 0.80$ m and $\alpha_0 \simeq 0.92$ Np/m. Thus it may be

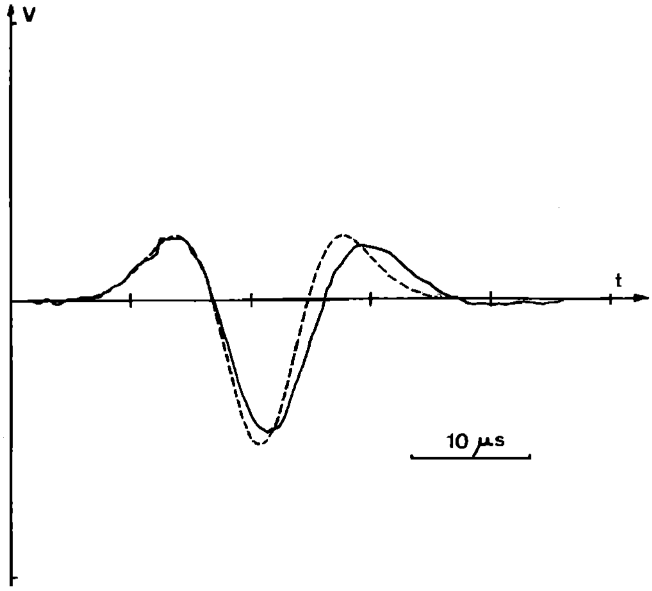

FIG. 3. Parametric signal. Primary signal as shown in Fig. 2, SL = 217 dB ref. 1 μPa (rms) — 1m. Solid line: recorded signal; dashed line: theoretical result.

assumed that condition (34) is satisfied as soon as $\not{h} > 14$ (then, $\not{h}\alpha_0\mathscr{R}_0 > 10$); that is to say, $\nu_{\mathrm{BF\,max}} < 440$ kHz. Measurements are made with $z \simeq 2.5$ m, this distance being far enough from the secondary sources (i.e., primary field) since $2\alpha_0 z \cong 20$ dB, but too short compared with $\mathscr{R}_0$ to deduce valid conclusions from off-axis records.

The power amplifier would allow the emitted pressure to be raised to 230 dB ref. 1 μPa — 1 m, which corresponds to the values of the ratio Γ of the attenuation length $\ell_a \cong 1$ m over the discontinuity length ℓ_d up to 15, assuming a perfect collimation of a pure harmonic primary beam.

Using a Gaussian modulation function (Fig. 2), the theoretical expression (32) is compared with the experimental on-axis result in Fig. 3. The observed discrepancy is more apparent when one uses a chirp-modulated beam (Fig. 4) to produce the result obtained also on-axis (Fig. 5). The demodulated signal should admit a quasiparabolic envelope, but, in fact, it exhibits a saturation effect for modulation frequencies over about 220 kHz. Although Fig. 5 was obtained with a fairly high source level, i.e., 224 dB ref. 1

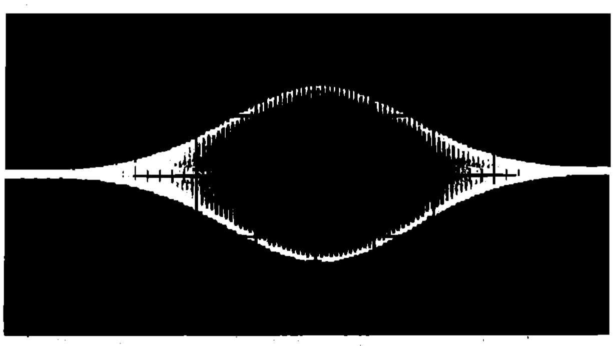

FIG. 2. Primary signal ($\nu_0 = 6.2$ MHz); $v_0(t) = \exp(-at^2)$, with $a = 1.54 \times 10^{10}\ \mathrm{s}^{-2}$.

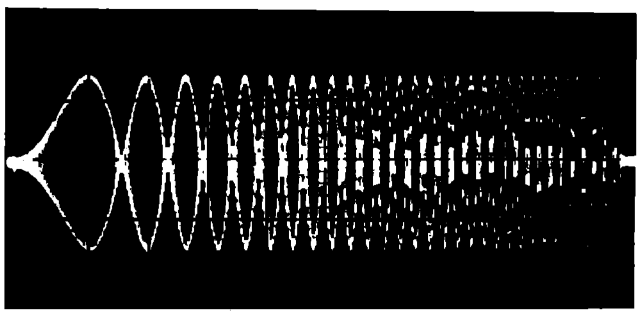

FIG. 4. Primary signal ($\nu_0 = 6.2$ MHz); $v_0(t) = \sin(at^2)$, with $a = 2.31 \times 10^{10}\ \mathrm{s}^{-2}$ and $0 \leqslant t \leqslant 64.5\ \mu$s.

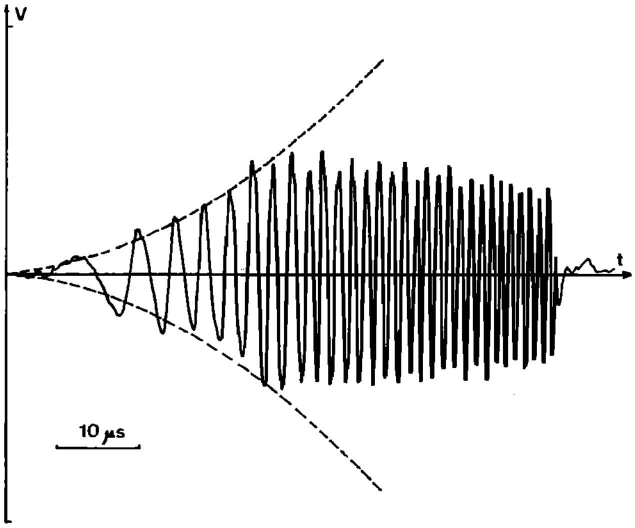

FIG. 5. Parametric signal. Primary signal as shown in Fig. 4, SL = 224 dB ref. 1 μPa (rms) − 1 m. Solid line: recorded signal; dashed line: theoretical result.

μPa − 1 m (rms) ($\Gamma \simeq 7$), it can be seen in Fig. 6 (SL = 230 dB; $\Gamma \simeq 15$) that this signal keeps its general shape for a large range of the maximum emitted pressure. The main difference between Figs. 5 and 6 is an increase of distortion in the demodulated signal. This phenomenon can be easily attributed to the parametric interactions between components of the harmonic bands issued from the primary beam. But this explanation is not quite satisfactory to justify the saturation effect above 220 kHz, which may result from an experimental artifact.

IV. CONCLUSION

The paraxial secondary farfield, created by the self-demodulation of a primary beam in which the space-time spectrum is narrow, has been established within a model of nonlinear weak interactions and using Fourier formalism. This theoretical result is a temporal convolution of the response

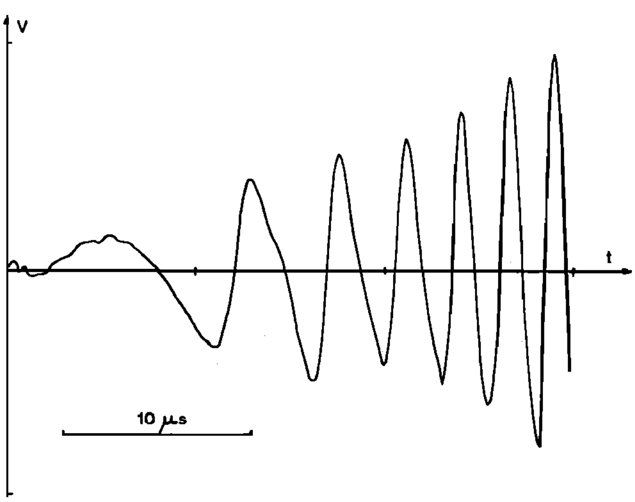

FIG. 6. Parametric signal.

[Eq. (32)] given by Berktay, with another function that depends upon the transducer shape and the structure of the interaction zone. With the help of further assumptions, even less restrictive than those of Stepanishen and Koenigs,[17] this second function can be calculated. In agreement with these authors, this function is found to split into the convolution product of the impulse response of a line source with that of the transducer. This impulse response of the transducer appears generally to have a negligible influence on the final result.

Thus the signal pattern is obtained on-axis in a straightforward manner, and a quite easy numerical computation gives it when the directions of observation diverge little from the beam axis. The complexity of this calculation increases heavily when these angles become greater, but the high parametric directivity removes any practical interest from this problem.

Certain problems remain without solution: (1) It is difficult to evaluate the secondary field in the complementary zone of the Fraunhofer area. (2) It may be asked whether any handy solution exists under very high nonlinear operating conditions. (3) The origin of the strong discrepancy between experimental and theoretical results is not quite understood when the parametric frequency ratio decreases.

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