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On Passive Symmetrical Two-Ports, Impedance Conversion and Power Transfer

Pierre Cervenka[1,2), Jacques Marchal[2,1)
1) CNRS, UMR 7190, Institut Jean le Rond d'Alembert, 78210 Saint-Cyr-l'Ecole, France.
pierre.cervenka@upmc.fr
2) UPMC Univ Paris 06, UMR 7190, Institut Jean le Rond d'Alembert, 78210 Saint-Cyr-l'Ecole, France.
jacques.marchal@upmc.fr

Summary
The theory of linear passive symmetrical two-ports is reviewed. Many results of practical interest can be derived from the very compact analytical basis of this model. However, such derivations are scattered in the literature. In addition, the demonstrations do not always take advantage of the generality that can be obtained by avoiding implementations built on particular applications. A complete, self consistent analysis is presented here. Attention is focused on power transfer and the relations between the impedances at each port. The specific case of transmission lines is finally addressed.

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1. Introduction

The powerful principles of reciprocity were formulated long ago in the optical and acoustical domains by Von Helmholtz [1, 2] and Lord Rayleigh [3] although Lamb [4] recognized that a generalization of these theorems was already contained in the former Lagrange's work [5]. Lorentz [6] extended the domain of application to electromagnetism. Extending the scope of the reciprocity principle is always an active field of research (e.g. [7]). Among such works, Goedbloed [8] and Potton [9] give in-depth reviews about reciprocity. The conditions of validity of the reciprocity principle are also a critical issue [10]. When properly defined, most linear passive networks are reciprocal although the theoretical existence of a counter-example has been shown by Tellegen [11] with the gyrator (in the mechanical domain, the actual existence of a passive material that would allow to build such a system is still questioned). The formal analogy of the principle of reciprocity across different domains of physical phenomenon has been pointed out, in particular between electrical and mechanical systems [12, 13, 14, 15]. It justifies the interest of all general properties that can be derived from formal analysis.

Another large class of systems is defined by symmetry. The dependency of symmetry on reciprocity is not a straightforward issue and has been thoroughly addressed [16, 17, 18, 19]. Many physical systems exhibit both the reciprocity and symmetry properties. Hence the related theorems provide powerful tools for dealing with a large range of applications [20], addressing for example impedance measurement [21, 22, 23], modeling of tubes [24, 25], and propagation in layered media [26]. The underlying concern is often power transfer.

This paper addresses the systems that can be described by means of a passive linear symmetrical two-port black box. Most analytical developments are known but scattered in journals and textbooks. The authors think useful to present here a synthetic view of the main results that can be derived from the very few hypotheses that make the basis of this model. In addition, the power transfer coefficients as expressed with non dimensional parameters in section 3, as well as the precise conditions that validate the approximations presented in section 3.5 have not been found elsewhere in the literature. We are primarily interested in the energy transfer through such a system, and in the impedance conversion that its presence involves. The formalism is presented without any explicit reference to propagation equations that would be derived from the physical phenomenon actually involved. The analysis is performed in the context of the harmonic steady state.

The black box that represents the system is passive, which means that it does not contain any internal source. Energy is exchanged with the external world in a way which can be described at each interface by a pair of state variables: a forcing, extensive variable V, and a response, intensive variable P. Their product represents the power fed into or pouring out the system via each terminal. The system is a linear two-port: there are two interfaces with a single pair of scalar conjugate variables at each of them; linear relations link these variables. Hence the scope of this paper does not extend to mechanical sys-

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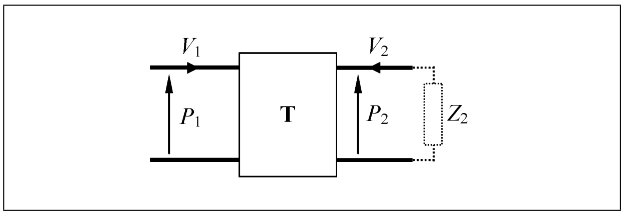

Figure 1. Electrical scheme of a two-port.

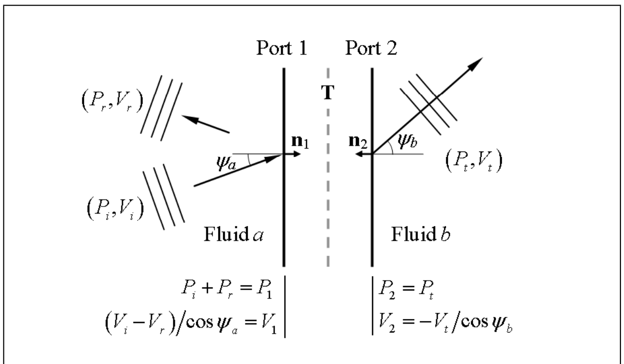

Figure 2. Simple acoustical example of a two-port.

tems that involve generalized forces and displacements because it would require 6 ports per concerned interface [27]. We consider a symmetric system, i.e. the black box remains the same when both ports are permuted. Because of the symmetry, the system is necessarily homogeneous, i.e. both interfaces have the same type (e.g. mechanical-mechanical or electrical-electrical). In the acoustic domain, a large variety of configurations can be taken into account, provided a pair of acoustic pressure and normal velocity defines the conditions at each interface. With an electrical system, P and V stand for voltages and currents, respectively (Figure 1). Note that symmetry implies reciprocity when dealing with a linear passive two-port.

For example, the system can be a multilayered symmetric media (solids and/or fluids), whose exterior faces are parallel planes in contact with outside fluids. The definition of the conjugate variables calls here for the time-angular spectra approach (time and spatial Fourier formalism). Considering the invariance of the time and spatial frequencies in the decomposition of the pressure and normal velocity fields at each interface, P and V stand for such spectral components at given time and spatial frequencies. In the simplest situation depicted Figure 2, Port 1 and Port 2 are in contact with semi-infinite homogeneous fluids. There is an incident plane wave (P_i, V_i) at Port 1 that is reflected (P_r, V_r), and a transmitted wave (P_t, V_t) out of Port 2. The invariance in the frequencies translates here through the Snell-Descartes relation between θ_a and θ_b. Note that $P_{1,2}$ and $V_{1,2}$ refer to the conditions at the interfaces. Note also that if the fluid b is not semi-infinite nor homogeneous, the load condition at Port 2 is modified because the conditions (P_2, V_2) at the interface will take into account an incoming, reflected wave. In case the direction of the plane waves is normal to the interfaces (1-D propagation), the restriction in the nature of the bounding media is withdrawn. A typical example arises when studying the power transfer between a transducer and the propagating media through a matching layer. Amongst other examples of application in the acoustic domain, the system can be also a waveguide so that the conjugate variables are defined with respect to any given mode.

The matrices that characterize a passive symmetrical two-port are derived in section 2.1. The interpretation in terms of one-way modes, as well as the introduction of the characteristic impedance, are thus straightforward. The relation between the impedances at both interfaces is analyzed in section 2.2. Classical limit cases are pointed out, in association with the propagation terminology (half/quarter/one- eighth wavelength lines). Power transfer is thoroughly addressed in section 3. The analysis is first conducted with respect to the input impedance. The power transfer coefficient exhibits a characteristic pattern in its dependency on the parameters of the two-port. This result does not seem to be known. The optimization of the power transfer within different situations is then discussed. In many particular problems, the parameters that describe the media supporting the system do not depend on the dimension that links the interfaces. Several results displayed in the previous sections are then furthermore expanded in section 4 when this property applies.

2. Passive symmetrical two-port

2.1. Transfer matrix

2.1.1. Two-port

Port 1 and Port 2 are referred as input and output, respectively. Note the convention of sign at the terminals in Figure 1. It implies that power entering the system is counted positive. For an acoustical system, it means that the vectors normal to the interfaces used to define the normal velocities are oriented inward the system at both ports (e.g. see Figure 2). Using this convention, the following impedances are defined according to

$$Z_1 = P_1 V_1^{-1}, \qquad Z_2 = -P_2 V_2^{-1}. \tag{1}$$

In the acoustical frame, Z defines a surface impedance because V refers to the spectral component of a normal velocity. In equation (1), Z_2 can be interpreted as the impedance that loads the network, whereas Z_1 stands for this load as seen from the input interface. If Z_2 is an actual impedance load, then $\Re e\{Z_2\} \geq 0$. Note that if the exterior media that stands next to Port 2 is an homogeneous, semi-infinite fluid devoid of any source, Z_2 is simply the characteristic impedance of this media. Furthermore considering a source applied at Port 1, the pressure P_1 and the velocity V_1 at this interface cannot be independently imposed, their ratio Z_1 being dictated by Z_2 as seen through the blackbox. One chooses to consider the source at the input side, i.e. Port 1, and the radiating media at the output side, i.e. Port 2. The converse configuration could be indeed considered, i.e. source at Port 2 with surface impedance Z_1 at Port 1. Because of the linearity, the general case with sources on both sides can be then handled merely by superposition.

Hence considering that there is no external source at Port 2, let us denote Z_T the shunt impedance and Y_T the open admittance as seen from Port 1, i.e.

$$\begin{aligned} Z_T &= \left(P_1 V_1^{-1}\right)\big|_{P_2=0} = Z_1(Z_2 = 0), \\ Y_T &= \left(P_1^{-1} V_1\right)\big|_{V_2=0} = Z_1^{-1}(Z_2^{-1} = 0). \end{aligned} \quad (2)$$

In the electrical transmission line language, it corresponds to measurements on the short or open circuited "stubs". In the acoustical field, Z_T is the input impedance seen when the output impedance is very weak; the admittance Y_T is the input admittance seen when the output impedance is very large. In both cases, the interface at Port 2 in these idealized experiments is close to a perfect mirror.

The power that is dissipated in a passive two-port cannot be negative, so that the real parts of Z_T and Y_T are positive,

$$\Re e\{Z_T\} > 0 \quad \text{and} \quad \Re e\{Y_T\} > 0. \quad (3)$$

Note that this property proceeds from a thermodynamic law and is always true. It does not depend on external conditions as it is for example the case with the sign of $\Re e\{Z_2\}$ or $\Re e\{Z_1\}$. Using (2), the transfer matrix $\boldsymbol{T}$ that describes the two-port reads

$$\begin{bmatrix} P_1 \\ V_1 \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} P_2 \\ -V_2 \end{bmatrix} \quad (4)$$

$$\text{with} \quad \boldsymbol{T} = \begin{bmatrix} a & bZ_T \\ aY_T & b \end{bmatrix}, \quad (5)$$

a and b being two complex constants.

Equation (4) can be also reordered in terms of impedance or admittance relations according to

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \boldsymbol{Z} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \boldsymbol{Y} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \quad (6)$$

$$\text{with} \quad \boldsymbol{Z} = Y_T^{-1} \begin{bmatrix} 1 & b(1 - Z_T Y_T) \\ a^{-1} & a^{-1} b \end{bmatrix}$$

$$\text{and} \quad \boldsymbol{Y} = \boldsymbol{Z}^{-1} = Z_T^{-1} \begin{bmatrix} 1 & -a(1 - Z_T Y_T) \\ -b^{-1} & ab^{-1} \end{bmatrix}. \quad (7)$$

2.1.2. Reciprocity

The straightforward definition of reciprocity is the symmetry of the impedance $\boldsymbol{Z}$ (or admittance $\boldsymbol{Y}$) matrix. Reciprocity implies that the determinant of the transfer matrix is equal to unit. In the particular case of a two-port, these properties are even equivalent, as well as the following particular "reciprocity relation"

$$V_1(P_1 = 0, P_2 = P) = V_2(P_1 = P, P_2 = 0). \quad (8)$$

Hence any of these definitions lead to

$$ab\left(1 - Z_T Y_T\right) = 1. \quad (9)$$

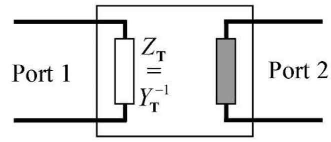

Figure 3. Equivalent electrical scheme of a reciprocal two-port when $Z_T = Y_T^{-1}$. Port 1 and Port 2 are isolated from each other.

A degenerated case arises when $Z_T = Y_T^{-1}$: there is always $Z_1 = Z_T$; both sides of the network are no longer related. There is something inside the black box that hides one port to the other (e.g. perfect reflector). The transfer matrix $\boldsymbol{T}$ is not defined. This case can be depicted as in Figure 3. For now, we consider the general case $Z_T Y_T \neq 0$.

Because of (9), the transfer matrix that describes a passive reciprocal two-port depends only on three complex parameters (a, Z_T and Y_T),

$$\boldsymbol{T} = \begin{bmatrix} a & a^{-1} Z_T (1 - Z_T Y_T)^{-1} \\ aY_T & a^{-1}(1 - Z_T Y_T)^{-1} \end{bmatrix} \quad (\Rightarrow \|\boldsymbol{T}\| = 1). \quad (10)$$

Equivalently, the impedance and admittance matrices read

$$\boldsymbol{Z} = Y_T^{-1} \begin{bmatrix} 1 & a^{-1} \\ a^{-1} & a^{-2}(1 - Z_T Y_T)^{-1} \end{bmatrix}$$

$$\text{and} \quad \boldsymbol{Y} = Z_T^{-1} \begin{bmatrix} 1 & -a(1 - Z_T Y_T) \\ -a(1 - Z_T Y_T) & a^2(1 - Z_T Y_T) \end{bmatrix}. \quad (11)$$

2.1.3. Symmetry

For a symmetrical system, it would not matter which port is the input port and which the output port. Hence the impedance $\boldsymbol{Z}$ and admittance $\boldsymbol{Y}$ matrices (7) have equal diagonal and antidiagonal elements: $e_{11} = e_{22}$ and $e_{12} = e_{21}$. Consequently, a symmetrical two-port is also necessarily reciprocal because the matrices $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are then symmetrical. Hence (9) must hold, and the equality of the diagonal terms in (11) implies in addition

$$a^2 = \left(1 - Z_T Y_T\right)^{-1}. \quad (12)$$

The transfer matrix (10) is reduced to

$$\boldsymbol{T} = a \begin{bmatrix} 1 & Z_T \\ Y_T & 1 \end{bmatrix}, \quad (13)$$

and the impedance and admittance matrices read

$$\boldsymbol{Z} = Y_T^{-1} \begin{bmatrix} 1 & a^{-1} \\ a^{-1} & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{Y} = Z_T^{-1} \begin{bmatrix} 1 & -a^{-1} \\ -a^{-1} & 1 \end{bmatrix}. \quad (14)$$

A passive symmetrical two-port is thus characterized by a transfer matrix which depends only on two parameters (Z_T, Y_T), whose determinant is unit and diagonal terms are equal. Note that the only knowledge of Z_T and Y_T is not sufficient to select the correct determination of the root of a^2 in (12). Additional information is needed to solve this problem: at a very low frequency, it can be for instance the sign of $\Re e\{P_1 P_2^*\}$ observed when $V_2 = 0$ which dictates the sign of the real part of a. This seed being set, complementary techniques such as phase unfolding in the frequency domain can be deployed.

The transfer matrix $\boldsymbol{T}$ can be diagonalized according to $\boldsymbol{T} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$ with

$$\boldsymbol{U} = \begin{bmatrix} Z_c & Z_c \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{D} = \begin{bmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{bmatrix}, \tag{15}$$

in which appears the characteristic impedance

$$Z_c = \sqrt{Z_T/Y_T}, \tag{16}$$

and the term ζ

$$\zeta = \left(\frac{1-\sqrt{Z_T Y_T}}{1+\sqrt{Z_T Y_T}} \right)^{1/2}. \tag{17}$$

These results can be checked in writting

$$\boldsymbol{T} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1} = \frac{1}{2}\begin{bmatrix} \zeta^{-1}+\zeta & (\zeta^{-1}-\zeta)Z_c \\ (\zeta^{-1}-\zeta)Z_c^{-1} & \zeta^{-1}+\zeta \end{bmatrix} = a\begin{bmatrix} 1 & Z_T \\ Y_T & 1 \end{bmatrix}. \tag{18}$$

Thorough the entire paper, the operator $\sqrt{..}$ stands as usual for the root determination whose real part is positive. The ambiguity addressed previously about the coefficient a is now deferred to the root that defines ζ in (17). This term is written in an exponential form by introducing the complex factor ϕ_T,

$$\begin{aligned} & \zeta = \exp\left(-\phi_T\right) \\ \Leftrightarrow \quad & \tanh\left(\phi_T + \mathrm{j}k\pi\right) = \sqrt{Z_T Y_T} \\ \Leftrightarrow \quad & \zeta = (-1)^k \exp\left(-\operatorname{arctanh}\sqrt{Z_T Y_T}\right). \end{aligned} \tag{19}$$

The tanh function being periodic modulo π with its imaginary argument, the ambiguity is now made explicit through the integer k. With this notation, the transfer matrix reads

$$\boldsymbol{T} = \begin{bmatrix} \cosh\phi_T & Z_c \sinh\phi_T \\ Z_c^{-1}\sinh\phi_T & \cosh\phi_T \end{bmatrix}, \tag{20}$$

which comes closer than (13) to the well known transmission line relation.

From the definition of the eigenvectors $[Z_c, \pm 1]^t$ which build the matrix $\boldsymbol{U}$ in (15), the output impedances $\pm Z_c$ are left unchanged after the transform $\boldsymbol{T}$,

$$Z_2 = \pm Z_c \;\Leftrightarrow\; Z_1 = \pm Z_c \tag{21}$$

(same sign in each equation). These eigenvectors and values can be interpreted in terms of wave propagation as two progressive waves that travel towards opposite directions,

$$\begin{aligned} & \boldsymbol{T}\begin{bmatrix} Z_c \\ 1 \end{bmatrix} = \zeta^{-1}\begin{bmatrix} Z_c \\ 1 \end{bmatrix} \Rightarrow \\ & \begin{bmatrix} P_1 \\ V_1 \end{bmatrix} = V\begin{bmatrix} Z_c \\ 1 \end{bmatrix}_{\text{Port1}} \rightarrow \begin{bmatrix} P_2 \\ -V_2 \end{bmatrix} = \zeta V\begin{bmatrix} Z_c \\ 1 \end{bmatrix}_{\text{Port2}}, \\ & \boldsymbol{T}\begin{bmatrix} Z_c \\ -1 \end{bmatrix} = \zeta\begin{bmatrix} Z_c \\ -1 \end{bmatrix} \Rightarrow \\ & \begin{bmatrix} P_1 \\ V_1 \end{bmatrix} = \zeta V\begin{bmatrix} Z_c \\ -1 \end{bmatrix}_{\text{Port1}} \leftarrow \begin{bmatrix} P_2 \\ -V_2 \end{bmatrix} = V\begin{bmatrix} Z_c \\ -1 \end{bmatrix}_{\text{Port2}}. \end{aligned} \tag{22}$$

In other words, any condition at the output can be split into two projections upon these eigenvectors,

$$\begin{bmatrix} P_2 \\ -V_2 \end{bmatrix} = \frac{P_2}{Z_c(1+\chi_2)}\left(\begin{bmatrix} Z_c \\ 1 \end{bmatrix} + \chi_2\begin{bmatrix} Z_c \\ -1 \end{bmatrix}\right), \tag{23}$$

$$\text{with} \quad \chi_2 = \frac{Z_2 - Z_c}{Z_2 + Z_c}, \tag{24}$$

which gives back to the input side:

$$\begin{bmatrix} P_1 \\ V_1 \end{bmatrix} = \frac{P_2}{Z_c(1+\chi_2)}\left(\zeta^{-1}\begin{bmatrix} Z_c \\ 1 \end{bmatrix} + \chi_2\zeta\begin{bmatrix} Z_c \\ -1 \end{bmatrix}\right). \tag{25}$$

The eigenvalue ζ (17) is the propagation operator that transforms a wave traveling from one interface to the other, whereas the inverse value ζ^{-1} in (25) accounts for retropropagation, i.e. finding back the wave at the interface from where it originated. Considering that there is a source only on the Port 1 side, the term χ_2 in (24) can be interpreted as the reflection coefficient observed at Port 2. Equivalently the ratio between the forward and backward waves at Port 1 is derived from (25),

$$\chi_1 = \chi_2\zeta^2 = \frac{Z_1 - Z_c}{Z_1 + Z_c}. \tag{26}$$

The magnitudes $|\chi_1|$ and $|\chi_2|$ differ on both sides only because of the losses encountered in the black box, which are taken into account by the real part of ϕ_T. Hence because of the convention stated above and the notation given in (19), that real part is always positive.

2.2. Characteristic impedance, input and output impedances

Looking now at the relation between Z_1 and Z_2 through the transfer matrix (13), there is

$$Z_1 = \frac{Z_2 + Z_T}{1 + Z_2 Y_T}, \qquad Z_2 = \frac{Z_1 - Z_T}{1 - Z_1 Y_T}. \tag{27}$$

It is easy to derive again directly from (27) that the characteristic impedance Z_c is the remarkable load value at the output side that is seen unchanged at the input side. Whenever Z_1 and Z_2 depart from this value, (27) does not give a convenient formula that links explicitly the differences $Z_2 - Z_c$ and $Z_1 - Z_c$. Nonetheless there is

$$Z_1 Z_2 = Z_c^2\left(1 + \frac{Z_2 - Z_1}{Z_c\sqrt{Z_T Y_T}}\right). \tag{28}$$

(28) allows to point out three particular cases that depend on the order of magnitude of $\sqrt{Z_T Y_T}$. With the help of (17), (19), there is

Case 1

$$\left|\sqrt{Z_T Y_T}\right| \ll 1 \quad \Rightarrow \quad \begin{cases} Z_1 \approx Z_2 \\ \zeta^2 \approx 1 \end{cases} \tag{29}$$

$$\Rightarrow \quad \phi_T = \mathrm{j}k\pi \quad \Rightarrow \quad \boldsymbol{T} = (-1)^k \boldsymbol{I}, \quad (k \in \mathbf{Z}) \tag{30}$$

in which $\boldsymbol{I}$ denotes the 2×2 identity matix. Because $|\arg(\zeta)| = 0 \bmod \pi$, this case is related to halfwavelength lines. The input impedance is the same as the output impedance, although being not equal to the characteristic impedance Z_c. It implies important practical consequences. This case is thoroughly investigated in section 3.5.

Case 2

$$\left|\sqrt{Z_T Y_T}\right| \gg 1 \quad \Rightarrow \quad \begin{cases} Z_1 Z_2 \approx Z_c^2 \\ \zeta^2 \approx -1 \end{cases} \tag{31}$$

$$\Rightarrow \quad \phi_T = \mathrm{j}\pi\left(\frac{1}{2} + k\right) \tag{32}$$

$$\Rightarrow \quad \boldsymbol{T} = (-1)^k \mathrm{j} \begin{bmatrix} 0 & Z_c \\ Z_c^{-1} & 0 \end{bmatrix}, \quad (k \in \mathbf{Z})$$

The propagation from one port to the other involves a phase rotation of a quarter of a circle ($\arg(\zeta) = \pi/2 \bmod \pi$), which is the signature of the so-called quarter-wavelength lines. The matrix $\boldsymbol{T}$ can be also interpreted as the transfer matrix of a gyrator with gyration impedance Z_c connected in cascade with a $\pm\pi/2$ phase shifting network [28]: it transfers a weak impedance into a very large impedance and vice versa. Less drastically, a $\lambda/4$ system can be used to perform a conversion between different impedances. For instance in the acoustic domain, a $\lambda/4$ layer with impedance Z_c is used to match the low impedance Z_2 of a propagating medium with the high impedance $Z_1 \approx Z_c^2 Z_2^{-1}$ of a transducer ceramic.

Case 3

Whenever the characteristic impedance Z_c is real, a remarkable property arises at the transition point for which the norm $|Z_T Y_T|$ is exactly equal to unit

$$\left|Z_T Y_T\right| = 1 \ \text{ and } \ Z_c \in \mathbf{R} \tag{33}$$

$$\Rightarrow \quad \begin{cases} \forall Z_2 \in \mathbf{R}, |Z_1| = Z_c, \\ \text{if } |Z_2| = Z_c, \forall \arg\{Z_2\}, Z_1 \in \mathbf{R} \end{cases}$$

Whatever is the value of a purely resistive load, the norm of the input impedance is equal to the norm of the characteristic impedance. Conversely, whenever the norm of the load is equal to the norm of the characteristic impedance, the input impedance is real. The propagation coefficient and the transfer matrix can be derived without assuming that Z_c is real:

$$\sqrt{Z_T Y_T} = \exp\left(\mathrm{j}\theta\right) \quad \Rightarrow \quad \zeta^2 = -\mathrm{j}\tan(\theta/2) \tag{34}$$

$$\Rightarrow \quad \phi_T = -\frac{1}{2}\ln\left(\tan\frac{|\theta|}{2}\right) + \mathrm{j}\left(\operatorname{sgn}(\theta)\frac{\pi}{4} + k\pi\right),$$

$$\Rightarrow \quad \boldsymbol{T} = \frac{1}{2}(-1)^k \sqrt{\frac{\mathrm{j}}{\sin\theta}} \tag{35}$$

$$\cdot \begin{bmatrix} \exp(-\mathrm{j}\theta/2) & Z_c \exp(\mathrm{j}\theta/2) \\ Z_c^{-1}\exp(\mathrm{j}\theta/2) & \exp(-\mathrm{j}\theta/2) \end{bmatrix},$$

($k \in \mathbf{Z}$). Note: $\theta \neq 0$ because $Z_T \neq Y_T^{-1}$ as stated in section 2.1.2.

This situation is known [29] as a $\lambda/8$ impedance transformer ($|\arg(\zeta)| = |\Im m\{\phi_T\}| = \pm\pi/4 \bmod \pi$).

3. Power transfer

3.1. Notations

We study here the power transfer from a source at the input of a passive linear symmetrical two-port, whose transfer matrix is given in (13), towards an impedance Z_2 that loads the output. The definition and meaning of Z_2 has been already discussed at the beginning of section 2. With electrical systems or acoustical waveguides, the source supplies energy per time unit. With systems that involve plane interfaces (and therefore angular spectra), the source delivers energy per time unit and per surface unit. In the following, we use the terminology "power" for both meanings.

The source delivers the power W_1 through the input port

$$W_1 = \frac{1}{2}\Re e\left\{P_1 V_1^*\right\} = \frac{1}{2}|P_1||V_1|\cos\theta_1 \tag{36}$$
$$\text{with} \quad \theta_1 = \arg\{Z_1\}.$$

(X^* denotes the complex conjugate of X).

Let us recall that with acoustic systems, (P_1, V_1) refers to the conditions at the input interface. Hence in the example given with Figure 2, W_1 should not be confused with the power associated with the incoming incident wave (P_i, V_i).

The power W_2 that is dissipated in the load Z_2 reads

$$W_2 = \tag{37}$$
$$\frac{|1+Z_T Y_T^*||P_1||V_1|\cos\delta_1 - \Re e\{Y_T\}|P_1|^2 - \Re e\{Z_T\}|V_1|^2}{2|1 - Z_T Y_T|}$$

$$\text{with} \quad \delta_1 = \theta_1 - \arg\{1 + Z_T Y_T^*\}. \tag{38}$$

The power delivered by the source and the power dissipated by the load are related through

$$W_2 = \frac{\mu}{2}|P_1||V_1| = \eta W_1 \tag{39}$$
$$\text{with} \quad \mu = \eta\cos\theta_1 \leq \eta < 1.$$

The coefficient η gives the relative amount of power that is actually transferred from the source to the load: the two-port dissipates the ratio $1-\eta$ of the total power delivered by the source. The coefficient μ characterizes the fraction of power that is dissipated in the load compared to the maximal power $|P_1||V_1|/2$ that the source can potentially deliver with the amplitudes $|P_1|$ and $|V_1|$:

$$\mu = \frac{|1 - Z_T Y_T|\Re e\{Z_2\}}{|1 + Z_2 Y_T||Z_2 + Z_T|} \tag{40}$$
$$= \frac{\Re e\left\{(Z_1 - Z_T)(1 - Z_1^* Y_T^*)\right\}}{|1 - Z_T Y_T||Z_1|}.$$

3.2. Optimal input impedance

3.2.1. Optimal phase

Given the amplitudes $|P_1|$ and $|V_1|$ at the input side, one seeks first at the phase θ_1 of the impedance Z_1 that maximizes the power W_2 dissipated in Z_2. Hence μ is maximal when $\delta_1 = 0$ (38), i.e.

$$\theta_{1\text{opt}} = \arg\left\{1 + Z_T Y_T^*\right\} \tag{41}$$

$$\text{i.e.} \quad Z_1 = |Z_1| \frac{1 + Z_T Y_T^*}{|1 + Z_T Y_T^*|}.$$

When this condition on the phase is true, the power that is dissipated in the load Z_2 reads

$$\begin{aligned} W_2 &= \frac{(R_{\min} + R_{\max})|P_1||V_1| - |P_1|^2 - R_{\min} R_{\max} |V_1|^2}{2(R_{\min} - R_{\max})} \\ &= \frac{R_{\max}}{R_{\min} - R_{\max}} \frac{|pv|}{2}, \end{aligned} \tag{42}$$

with

$$|p| = |P_1| - R_{\min}|V_1|, \quad |v| = |V_1| - R_{\max}^{-1}|P_1|$$

and

$$\begin{aligned} R_{\min} &= \frac{2\Re e\{Z_T\}}{|1 + Z_T Y_T^*| + |1 - Z_T Y_T|}, \\ R_{\max}^{-1} &= \frac{2\Re e\{Y_T\}}{|1 + Z_T Y_T^*| + |1 - Z_T Y_T|}. \end{aligned} \tag{43}$$

It can be checked that there is always $0 < R_{\min} < R_{\max}$. The locus of the solutions $(|P_1|, |V_1|)$ that lead to dissipate at best the given power W_2 in the load is the hyperbola with axes $|p| = 0$ and $|v| = 0$ (see Figure 4). All the physical solutions are necessarily located in the sector that is bounded by these two half lines: the power that is dissipated in the load cannot be negative. In other words, the real part of the load Z_2 being positive, the norm $|Z_1|$ is in the range

$$R_{\min} < |Z_1| < R_{\max}. \tag{44}$$

Let us introduce the geometrical mean R_{m} of the resistances $R_{\min}$ and $R_{\max}$,

$$R_{\text{m}} = \sqrt{R_{\min} R_{\max}} = \sqrt{\frac{\Re e\{Z_T\}}{\Re e\{Y_T\}}}. \tag{45}$$

Notice that R_{m} is not equal to Z_c. In order to browse the domain of variation of the norm $|Z_1|$ (44), one uses the scalar parameter σ that is defined in the domain $[0, 1]$ so that $|Z_1|(\sigma = 0) = R_{\min}$ and $|Z_1|(\sigma = 1) = R_{\max}$:

$$|Z_1|(\sigma) = R_{\min}^{1-\sigma} R_{\max}^{\sigma} = R_{\text{m}} \left(\frac{R_{\min}}{R_{\max}}\right)^{0.5-\sigma}, \tag{46}$$

with $0 < \sigma < 1$.

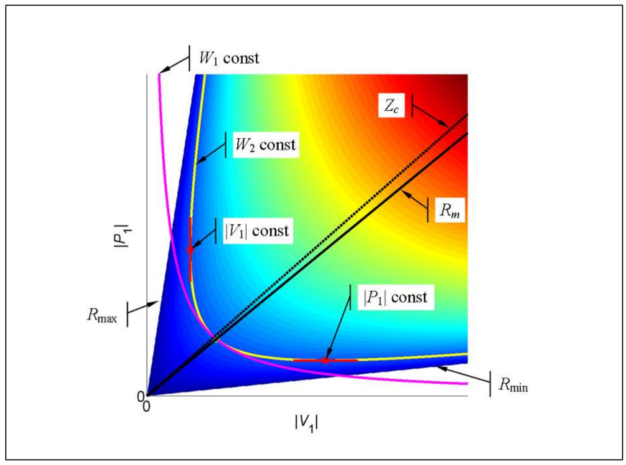

Figure 4. Power transfer optimization.

When the phase of Z_1 is optimal, the transfer coefficient μ is equal to

$$\begin{aligned} \mu_{\delta_1=0}(|Z_1|) &= \frac{(R_{\max} - |Z_1|)(|Z_1| - R_{\min})}{|Z_1|(R_{\max} - R_{\min})} \\ &= \frac{1}{\beta}\left(1 - \sqrt{1-\beta^2} \cosh\left((\sigma - 0.5) \ln \frac{1-\beta}{1+\beta}\right)\right), \end{aligned} \tag{47}$$

in which is introduced the coefficient β that characterizes the relative difference between the resistances $R_{\min}$ and $R_{\max}$,

$$\begin{aligned} 0 < \beta &= \frac{R_{\max} - R_{\min}}{R_{\max} + R_{\min}} = \left|\frac{1 - Z_T Y_T}{1 + Z_T Y_T^*}\right| < 1, \\ &\left(\frac{R_{\min}}{R_{\max}} = \frac{1-\beta}{1+\beta}\right). \end{aligned} \tag{48}$$

Whenever the orders of magnitude (29) or (31) hold, then the coefficient β tends towards unity, which is equivalent to $R_{\min} \ll R_{\max}$.

Figure 5 shows the mapping of $\mu_{\delta_1=0}(\beta, \sigma)$. One observes the following properties

$$\begin{aligned} &\mu_{\delta_1=0} \xrightarrow[\sigma \to 0 \text{ or } 1]{} 0, \qquad \mu_{\delta_1=0} \xrightarrow[\beta \to 0]{} 0, \\ &\mu_{\delta_1=0} \xrightarrow[\substack{\beta \to 1 \text{ with} \\ 0.5 - |\sigma - 0.5| \not\ll 1}]{} 1. \end{aligned} \tag{49}$$

The latter approximation close to the limit $\beta = 1$ should not hide the sharp evolution of $\mu(\beta)$, as well as the sharp variation of $\mu(\sigma)$ in the vicinity of the boundaries $\sigma = 0$ and $\sigma = 1$,

$$\left.\frac{\partial \mu_{\delta_1=0}}{\partial \beta}\right|_{0.5 - |\sigma - 0.5| \not\ll 1} \xrightarrow[\beta \to 1]{} +\infty. \tag{50}$$

From (47), the power transfer is maximal when

$$\sigma = 1/2, \quad \text{i.e.} \quad |Z_1|_{\text{opt}} = R_m. \tag{51}$$

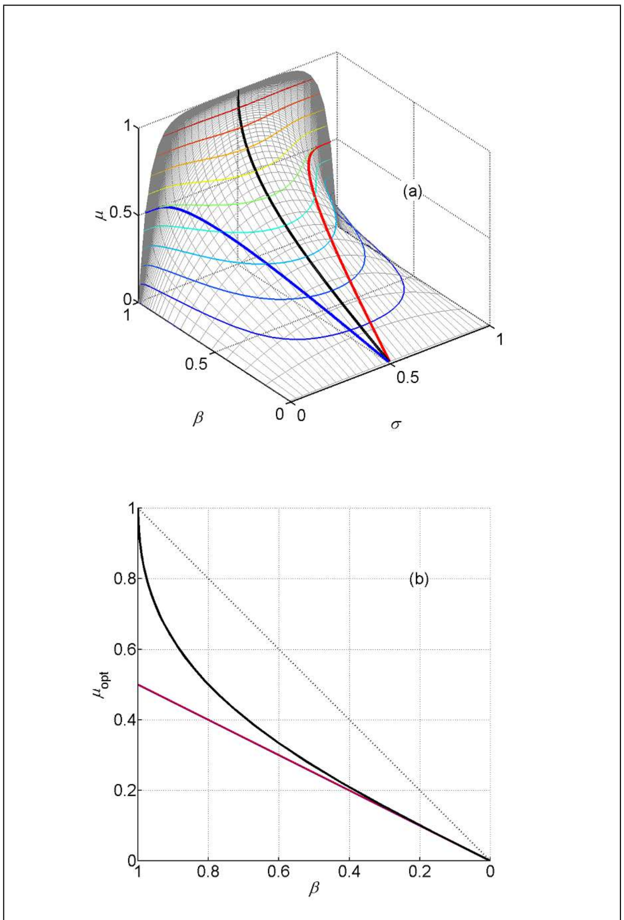

Figure 5. (a) $\mu(\beta,\sigma,\delta_1=0)$. Bold black: $\mu_{\text{opt}}=\mu(\sigma=1/2,\delta_1=0)$, bold red: $|P_1|$ constant, bold blue: $|V_1|$ constant. (b) Projection. Solid black: $\mu_{\text{opt}}(\beta)=\mu(\beta,\sigma=1/2,\delta_1=0)$, blue/red: $\mu_{\text{opt}}\big|_{\text{constant }|P_1|\text{ or }|V_1|}(\beta)=\beta/2$.

Given the dissipated power W_2, the power that is required at the source is thus minimal when the load Z_2 is equal to

$$
\begin{aligned}
Z_{2\text{opt}} &= \frac{Z_{1\text{opt}}-Z_T}{1-Z_{1\text{opt}}Y_T} \\
&= \frac{\Re e\{Z_T\}-\mathrm{j}\Im m\{Z_{1\text{opt}}\}}{\Re e\{Y_T Z_{1\text{opt}}\}} \\
&= \frac{\sqrt{\Re e\{Z_T\}\Re e\{Y_T\}}-\mathrm{j}\sin\theta_{1\text{opt}}}{\Re e\{Y_T\exp(\mathrm{j}\theta_{1\text{opt}})\}},
\end{aligned}
\tag{52}
$$

with

$$Z_{1\text{opt}}=R_m\exp(\mathrm{j}\theta_{1\text{opt}}). \tag{53}$$

The coefficient μ is then equal to (see Figure 5b)

$$
\begin{aligned}
\mu_{\text{opt}} &= \mu_{\delta_1=0}\big(|Z_{1\text{opt}}|\big)=\mu_{\delta_1=0}\big(\beta,\sigma=1/2\big) \\
&= \frac{\sqrt{R_{\max}}-\sqrt{R_{\min}}}{\sqrt{R_{\max}}+\sqrt{R_{\min}}}=\beta^{-1}\big(1-\sqrt{1-\beta^2}\big).
\end{aligned}
\tag{54}
$$

It can be noticed also that

$$\frac{\beta}{2}<\mu_{\text{opt}}=\frac{\beta}{1+\sqrt{1-\beta^2}}<\beta\le 1. \tag{55}$$

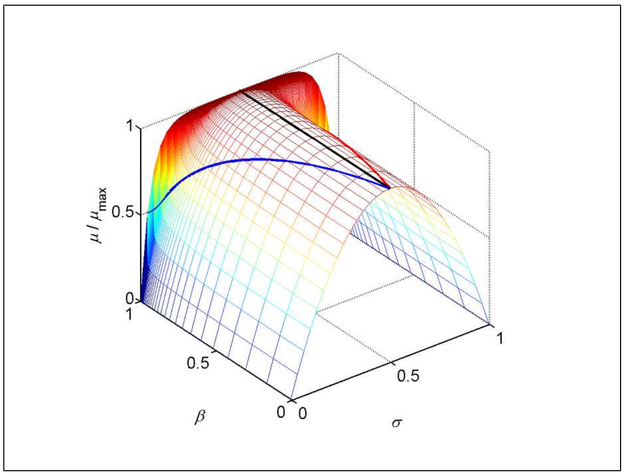

Figure 6. Relative power transfer coefficient $\xi=\mu/\mu_{\text{opt}}=\mu(\beta,\sigma)/\mu(\beta,1/2)$.

Figure 6 exhibits the coefficient μ relatively to the maximal values μ_{opt}, when the only phase is optimal,

$$
\begin{aligned}
\xi_{\delta_1=0}\big(\beta,\sigma\big) &= \frac{\mu_{\delta_1=0}}{\mu_{\text{opt}}} \\
&= \frac{1-\sqrt{1-\beta^2}\cosh\Big(\big(\sigma-0.5\big)\ln\frac{1-\beta}{1+\beta}\Big)}{1-\sqrt{1-\beta^2}}.
\end{aligned}
\tag{56}
$$

The following limits can be observed in this figure

$$\xi \xrightarrow[0.5-|\sigma-0.5|\not\ll 1]{\beta\to 1,} 1$$

and $$\xi \xrightarrow[\beta\to 0]{} \xi_0=1-4(\sigma-0.5)^2. \tag{57}$$

As for the function $\mu(\beta,\sigma)$, the straightforward limit of ξ when β is close to unit is misleading: there is still a sharp change in the vicinity of the boundaries $\sigma=0$ and $\sigma=1$. Also, $\xi(\beta)$ varies very rapidly. On the other hand, the ratio ξ is close to the function ξ_0 (57) as long as β remains not too close to unit. In addition there is always $\xi(\beta,\sigma)>\xi_0(\sigma)$, which shows that ξ_0 is a worse estimate of ξ. Consequently, the lack of transfer is smaller than -3 dB (half-power) as long as the norm of the input impedance remains in the range dictated by (46) with $\sigma=0.5\pm 0.5/\sqrt{2}$. When β is close to unit, the tolerance limits are larger, but there is no elementary approximation – e.g. a polynomial expression – to quantify this gain.

Let us recall that the optimal solution (52) can be reached only if the source at the input interface can supply the following amplitudes

$$
\begin{aligned}
|P_1| &= \sqrt{2|Z_{1\text{opt}}|\mu_{\text{opt}}^{-1}W_2}, \\
|V_1| &= \sqrt{2|Z_{1\text{opt}}|^{-1}\mu_{\text{opt}}^{-1}W_2},
\end{aligned}
\tag{58}
$$

in which $|Z_{1\text{opt}}|$ and μ_{opt} are given by (51) and (54).

3.2.2. General case (non optimal phase)

In the general case, i.e. when the phase is not optimal, the power that is dissipated by the load Z_2 (37) can still be written in terms of the previously defined parameters δ_1 (38), $R_{\min}$ and $R_{\max}$ (43), and β (48),

$$W_2 = \frac{(R_{\min}+R_{\max})|P_1||V_1|\cos\delta_1 - |P_1|^2 - R_{\min}R_{\max}|V_1|^2}{2(R_{\max}-R_{\min})} = \frac{r_{\max}}{R_{\max}-R_{\min}}\frac{|p'v'|}{2}, \tag{59}$$

with

$$|p'| = |P_1| - r_{\min}|V_1|, \quad |v'| = |V_1| - r_{\max}^{-1}|P_1|$$

and

$$r_{\min} = \frac{1+\beta}{\cos\delta_1 + \sqrt{\beta^2-\sin^2\delta_1}}R_{\min}, \quad r_{\max}^{-1} = \frac{1+\beta}{\cos\delta_1 + \sqrt{\beta^2-\sin^2\delta_1}}R_{\max}^{-1}. \tag{60}$$

The geometrical mean of $r_{\min}$ and $r_{\max}$ is the same as when the phase is optimal,

$$\sqrt{r_{\min}r_{\max}} = \sqrt{R_{\min}R_{\max}} = R_m. \tag{61}$$

The phase θ_1 of Z_1, and consequently the difference δ_1, are constrained by the impedance Z_2, which translates into the inequality

$$|\sin\delta_1| \le \beta \le 1. \tag{62}$$

Being given the phase difference δ_1, the locus of the solutions $(|P_1|, |V_1|)$ that lead to dissipate at best a given power W_2 in the load is the hyperbola whose axes are $|p'| = 0$ and $|v'| = 0$. The physical solutions lay in a sector bounded by two lines whose angle is now smaller than if the phase is optimal,

$$R_{\min} \le r_{\min} \le R_m \le r_{\max} \le R_{\max} \tag{63}$$

and

$$0 \le \frac{r_{\max}-r_{\min}}{R_{\max}-R_{\min}} = \sqrt{1-\beta^{-2}\sin^2\delta_1} \le 1.$$

The domain $[r_{\min}r_{\max}]$ of the norm $|Z_1|$ can be browsed again with the parameter σ which is now defined so that $|Z_1|(\sigma=0) = r_{\min}$ and $|Z_1|(\sigma=1) = r_{\max}$.

$$|Z_1|(\sigma,\delta_1) = r_{\min}^{1-\sigma}r_{\max}^{\sigma} = R_m\left(\frac{r_{\min}}{r_{\max}}\right)^{0.5-\sigma}, \tag{64}$$

with $0 < \sigma < 1$.

With these notations, the general expression of the transfer coefficient μ (40) reads

$$\mu = \frac{(r_{\max}-|Z_1|)(|Z_1|-r_{\min})}{|Z_1|(R_{\max}-R_{\min})} = \frac{1}{\beta}\left(\cos\delta_1 - \sqrt{1-\beta^2}\cosh\left((\sigma-0.5)\ln\frac{r_{\min}}{r_{\max}}\right)\right), \tag{65}$$

where

$$\frac{r_{\min}}{r_{\max}} = \frac{\cos\delta_1 - \sqrt{\beta^2-\sin^2\delta_1}}{\cos\delta_1 + \sqrt{\beta^2-\sin^2\delta_1}}. \tag{66}$$

This result can be compared with (47), in which in particular $R_{\min}/R_{\max}$ is given by (48). Given a constant module $|Z_1|$, the loss of efficiency because of the non optimal phase ($\delta_1 \ne 0$) can be quantified with

$$\mu_{\delta_1\ne 0}(|Z_1|) = \mu_{\delta_1=0}(|Z_1|) - 2\beta^{-1}\sin^2(\delta_1/2), \tag{67}$$

recalling that this latter expression is only meaningful as long as $r_{\min} < |Z_1| < r_{\max}$.

When the phase is not optimal, the maximal transfer is still obtained with the same norm $|Z_{1\text{opt}}| = R_m$ as when the phase is optimal (see equation 51). The transfer coefficient is

$$\mu_{\delta_1}(|Z_{1\text{opt}}|) = \beta^{-1}\left(\cos\delta_1 - \sqrt{1-\beta^2}\right). \tag{68}$$

to be compared with (54).

3.3. Optimal source at constant pressure or constant velocity

The previous developments are founded on the search for maximal transfer coefficients without any constraints on the amplitudes $|P_1|$ and $|V_1|$ at the input interface, but for their ratio. This section addresses the conditions that maximize the amount of power dissipated in the load, being given a constant amplitude $|V_1|$ or $|P_1|$ at the source. It comes back to look at the hyperbola that are tangent to the lines $V = |V_1|$ and $P = |P_1|$. The condition (41) about the phase of the input impedance is still assumed to be fulfilled ($\delta_1 = 0$).

Given a constant amplitude $|V_1|$, the configuration that leads to the maximal power dissipation in the load is derived by solving $\mathrm{d}W_2/\mathrm{d}P_1 = 0$. From (27), (42), there is

$$|Z_1| = \frac{1}{2}\left(R_{\min}+R_{\max}\right) \Rightarrow Z_1 = \frac{1+Z_TY_T^*}{2\Re e\{Y_T\}} \Rightarrow Z_2 = Y_T^{*-1}. \tag{69}$$

Accordingly given a constant amplitude $|P_1|$, the input impedance that leads to dissipate the maximal power in the load is obtained by solving $\mathrm{d}W_2/\mathrm{d}V_1 = 0$. From (27), (42), there is

$$|Z_1|^{-1} = \frac{1}{2}\left(R_{\min}^{-1}+R_{\max}^{-1}\right) \Rightarrow Z_1 = \frac{2\Re e\{Z_T\}}{1+Z_T^*Y_T} \Rightarrow Z_2 = Z_T^*. \tag{70}$$

In both cases, the transfer coefficient $\mu_{\delta_1=0}$ is reduced to

$$\mu_0 = \beta/2 = (1+\sqrt{1-\beta^2})\mu_{\text{opt}}/2 \ge \mu_{\text{opt}}/2. \tag{71}$$

Hence μ_0 is larger than half the optimal transfer coefficient. The solutions (69) and (70) correspond to the following locus of the parameters (σ, β),

$$\sigma^{-1} = \begin{cases} 1 - \dfrac{\ln(1+\beta)}{\ln(1-\beta)} & \text{at const. } |V_1|, \\ 1 - \dfrac{\ln(1-\beta)}{\ln(1+\beta)} & \text{at const. } |P_1|. \end{cases} \tag{72}$$

The curves that represent equations (71), (72) are drawn in red and blue in Figures 5–6.

When the orders of magnitude (31) or (29) apply, $\beta \approx 1$ so that $\mu_0 \approx 1/2$.

3.4. Input impedance equal to load

The equality of the impedances Z_1 and Z_2 with the characteristic impedance Z_c (21) does not satisfy *a priori* the criteria that optimize the power transfer. From (39), (40), the transfer coefficients read in that case

$$Z_1 = Z_2 = Z_c \;\Rightarrow \tag{73}$$
$$\begin{cases} \mu_c = \eta_c \cos(\arg\{Z_c\}), \\ \eta_c = \left| \dfrac{1-\sqrt{Z_T Y_T}}{1+\sqrt{Z_T Y_T}} \right| = \zeta^* \zeta = \exp\left(-2\Re e\{\phi_T\}\right) < 1. \end{cases}$$

When the orders of magnitude (31) or (29) apply, there is no significant power dissipation in the two-port,

$$\left.\begin{array}{r} \left|\sqrt{Z_T Y_T}\right| \gg 1 \\ \text{or } \left|\sqrt{Z_T Y_T}\right| \ll 1 \end{array}\right\} \text{ and } Z_{1,2} = Z_c \;\Rightarrow\; \eta_c = 1. \tag{74}$$

At the transition configuration (34), the transfer coefficient η reads

$$\left|\sqrt{Z_T Y_T}\right| = 1 \text{ and } Z_{1,2} = Z_c \;\Rightarrow \tag{75}$$
$$\eta_c = \left| \tan\left(\frac{1}{2}\arg\left\{\sqrt{Z_T Y_Z}\right\}\right)\right|.$$

In that case, there is no loss in the network if

$$\left|\arg\left\{\sqrt{Z_T Y_T}\right\}\right| \approx \pi/2,$$

which can be fulfilled only if Z_T and Y_T are both pure imaginary parameters (equation (3) and $Z_T Y_T \in \mathbf{R}^-$).

According to its definition, the characteristic impedance Z_c is real only if the phases of Z_T and Y_T are equal. When this condition is met, the optimal phase of Z_1 (41) is null ($Z_T Y_T^* \in \mathbf{R}$), and $Z_2 = Z_1 = Z_c$ is the optimal solution given by (45), (52), (53),

$$\arg\left\{Z_T\right\} = \arg\left\{Y_T\right\} \;\Rightarrow \tag{76}$$
$$Z_{2\text{opt}} = Z_{1\text{opt}} = Z_c \in \mathbf{R}^{+*}.$$

The expression of η_c is still given by (73), but Z_c is now real so that

$$\mu_c = \mu_{\text{opt}} = \eta_c = \eta_{\text{opt}}. \tag{77}$$

The efficiency of the source and the power transfer are close to perfect ($\mu = \eta = 1$) whenever (74), or (75) together with $Z_T Y_T \in \mathbf{R}^-$, applies.

3.5. Particular case $|Z_T| \ll |Y_T^{-1}|$

Several additional results can be derived in the particular case of practical interest defined by the following order of magnitude:

$$|Z_T Y_T| \ll 1. \tag{78}$$

Recalling that the real parts of Z_T and Y_T are both positive, it must be kept in mind in the derivations of this section that (3), (78) are equivalent to

$$0 < \left.\begin{array}{c} \Re e\left\{Z_T\right\} \\ \left|\Im m\left\{Z_T\right\}\right| \end{array}\right\} \le \left|Z_T\right| \tag{79}$$
$$\ll \left|Y_T^{-1}\right| \le \left\{\begin{array}{c} \Re e^{-1}\left\{Y_T\right\} \\ \left|\Im m^{-1}\left\{Y_T\right\}\right| \end{array}\right. .$$

It implies straightforward approximations on parameters introduced in section 3.2,

$$\theta_1 \approx \delta_1, \quad \beta \to 1, \quad \begin{cases} R_{\min} \approx \Re e\left\{Z_T\right\} \\ R_{\max}^{-1} \approx \Re e\left\{Y_T\right\} \end{cases}$$
$$\text{and} \quad \begin{cases} r_{\min} \approx R_{\min}/\cos\theta_1, \\ r_{\max}^{-1} \approx R_{\max}/\cos\theta_1, \end{cases}. \tag{80}$$

An important consequence is that the phase of Z_1 is optimal when being null. Also from the definition (16), the order of magnitude of the characteristic impedance Z_c is such that

$$\left|Z_T\right| \ll \left|Z_c\right| \ll \left|Y_T^{-1}\right|. \tag{81}$$

On the other hand, it has been already pointed out that β being close to unit does not necessarily imply obvious simplifications on the transfer functions.

The norms $|Z_1|$ or $|Z_2|$ cannot have simultaneously the same order of magnitude as $|Y_T|^{-1}$ and $|Z_T|$. In addition, (27) shows that $|Z_1|$ and $|Z_2|$ share the same order of magnitude relatively to $|Y_T|^{-1}$ and $|Z_T|$. In other words, one of the following non exclusive propositions is necessarily true, and depending on the case, the differences between impedances or between admittances, as well as the transfer coefficient μ (40) read

$$\left(\left|Z_1\right| \ll \left|Y_T^{-1}\right| \quad \Leftrightarrow \quad \left|Z_2\right| \ll \left|Y_T^{-1}\right|\right)$$
$$\Rightarrow \quad \left|\begin{array}{l} Z_1 - Z_2 \approx Z_T, \\ \mu = \cos\theta_1 - \left|Z_1\right|^{-1}\Re e\left\{Z_T\right\}, \end{array}\right. \tag{82}$$

or

$$\left(\left|Z_T\right| \ll \left|Z_1\right| \quad \Leftrightarrow \quad \left|Z_T\right| \ll \left|Z_2\right|\right)$$
$$\Rightarrow \quad \left|\begin{array}{l} Z_1^{-1} - Z_2^{-1} \approx Y_T, \\ \mu = \cos\theta_1 - \left|Z_1\right|\Re e\left\{Y_T\right\}. \end{array}\right.$$

In any instance, provided the impedance Z_1 is not essentially reactive ($\tan\theta_1 \not\gg 1$), the fraction of power that is dissipated in the network is given by

$$1 - \eta = \Re e^{-1}\left\{Z_1\right\}\Re e\left\{Z_T\right\} \tag{83}$$
$$+ \Re e^{-1}\left\{Z_1^{-1}\right\}\Re e\left\{Y_T\right\}.$$

The results stated in (82), (83) lead to interesting simplifications if additional conditions are introduced. Hence, when both orders of magnitude given in (82) are simultaneously met, the load as seen through the network is not significantly modified, although the common value can depart from the characteristic impedance Z_c. That is the result already pointed out in (29). In addition, still provided the impedance $Z_{1,2}$ is not essentially reactive, the coefficient μ can be directly estimated from the argument θ_2 of the load impedance (because $\theta_1 = \theta_2$), and the power that is dissipated in the network is negligible,

$$\left.\begin{array}{l} \left|Z_T\right| \ll \left|Z_{1,2}\right| \ll \left|Y_T\right|^{-1} \\ \text{and} \quad \left|\tan\theta_{1,2}\right| \not\gg 1 \end{array}\right\} \tag{84}$$
$$\Rightarrow \quad \left\{\begin{array}{l} Z_1 = Z_2, \\ \mu = \cos\theta_{1,2}, \\ \eta = 1. \end{array}\right.$$

In order to complete the discussion, it can be checked that when $|\theta_{1,2}|$ is close to $\pi/2$, there is still $Z_1 = Z_2$, the transfer coefficient μ tends toward zero, but η is not necessarily equal to unit.

Starting from (82), (83), another condition of great practical importance that is less restrictive than (84) and that still leads to a negligible relative amount of power dissipated in the network can be also derived,

$$\left.\begin{array}{l} \Re e\left\{Z_T\right\} \ll \left|Z_{1,2}\right| \ll \Re e^{-1}\left\{Y_T\right\} \\ \text{and} \quad \tan\theta_{1,2} \not\gg 1 \end{array}\right\} \tag{85}$$
$$\Rightarrow \quad \left\{\begin{array}{l} \mu = \cos\theta_1, \\ \eta = 1. \end{array}\right.$$

Compared to (84), the load Z_2 and the impedance Z_1 seen at the input are not necessarily equal, but the possible significant differences in the impedances $Z_{1,2}$ or admittances $Z_{1,2}^{-1}$ are an imaginary component

$$\tan\theta_{1,2} \not\gg 1 \quad \text{and} \quad \left\{\begin{array}{l} \Re e\left\{Z_T\right\} \ll \left|Z_{1,2}\right| \ll \left|Y_T\right|^{-1} \\ \quad \Rightarrow \quad Z_1 - Z_2 \approx \mathrm{j}\Im m\left\{Z_T\right\} \\ \text{or} \\ \left|Z_T\right|^{-1} \ll \left|Z_{1,2}\right| \ll \Re e^{-1}\left\{Y_T\right\} \\ \quad \Rightarrow \quad Z_1^{-1} - Z_2^{-1} \approx \mathrm{j}\Im m\left\{Y_T\right\}. \end{array}\right. \tag{86}$$

The only conditions (78) and (85) leave a large range in the choice of Z_1, which thus can differ significantly from the characteristic impedance Z_c, still avoiding power losses in the network. Nevertheless, the source does not necessarily work in an optimal regime because it provides only the ratio $\cos\theta_1$ of the total power that could be potentially delivered. For the source to work optimally, the input impedance Z_1 must be purely resistive,

$$\Re e\left\{Z_T\right\} \ll Z_1 = R \ll \Re e^{-1}\left\{Y_T\right\}$$
$$\Rightarrow \quad \mu = \eta \approx 1. \tag{87}$$

The condition on phase (41) is indeed reduced here to $\theta_1 = 0$. The main difference with the general case $|Z_T Y_T| \not\ll 1$ is that the constraint (51) about the norm is no longer needed to achieve an optimal transfer. (86) gives straightforwardly the imaginary component that the load impedance or admittance (Z_2 or Z_2^{-1}) must have for the input impedance to be purely resistive in (87). If the imaginary part of the load does not compensate exactly for the network influence, (86) can still be used to estimate the input phase θ_1, so that (85) gives the corresponding decrease in the transfer factor μ.

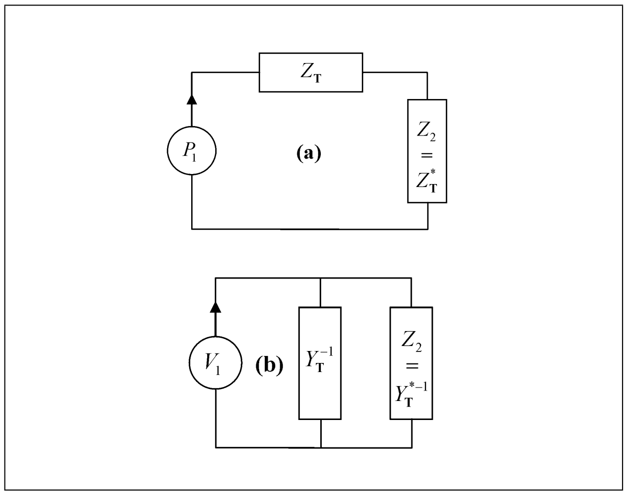

Figure 7. Equivalent electrical schemes of the optimal adaptation when $|Z_T Y_T| \ll 1$, at constant input pressure (a) and constant velocity (b).

Concerning the optimal solutions at constant input velocity or pressure, results given in section 3.3 are easily translated when (78) applies. The transfer coefficients μ and η are equal,

$$\mu = \eta = 0.5. \tag{88}$$

At constant input amplitude $|P_1|$, the optimal solution is reached with

$$Z_2 = Z_T^* \quad \Rightarrow \quad Z_1 = 2\Re e\left\{Z_T\right\}. \tag{89}$$

At constant input amplitude $|V_1|$, there is

$$Z_2 = Y_T^{*-1} \quad \Rightarrow \quad Z_1 = \left(2\Re e\left\{Y_T\right\}\right)^{-1}. \tag{90}$$

The solutions (89) and (90) are the very classical results shown in Figure 7.

4. Transmission line

4.1. Morphism

Let us consider a family of passive linear symmetrical two-ports $\{\boldsymbol{T}_x\}$ which realizes the following morphism from $(\mathbf{R}, +)$ into $(\{\boldsymbol{T}\}, \times)$:

$$\forall x, y \in \mathbf{R}, \qquad \boldsymbol{T}_{x+y} = \boldsymbol{T}_x \boldsymbol{T}_y. \tag{91}$$

This property appears in many applications where the real parameter x is a length. Equation (91) may apply whenever the parameters describing the media that supports the studied phenomenon do not depend on the x parameter, whereas a single pair of intensive and extensive variables (P_x, V_x) can be associated to each value of x. A classical example is the modeling of an electrical cable: the transfer matrix associated with a piece of the line is equal to the product of the matrices associated with all parts that build any partition of the initial piece. In the acoustical domain, other trivial examples are obtained when studying propagation between two planes (with distance x) in a homogeneous media or in a tube with constant cross section. The model is also valid whatever the dispersive character of the propagation is (e.g in waveguides).

Let us denote $Z_x = Z_T$ and $Y_x = Y_T$ the parameters associated with the matrix $\boldsymbol{T}_x$. The commutativity of (91) implies that the ratio Z_x/Y_x does not depend on the length x,

$$\forall x, \qquad Z_x/Y_x = Z_c^2. \tag{92}$$

That is why (16) can be said to define the characteristic impedance of a media when (91) holds.

In the decomposition (15), there is only the diagonal matrix $\boldsymbol{D}$ that depends on the length x. Hence the property (91) incorporates into this matrix according to

$$\forall x, y \in \mathbf{R}, \quad \boldsymbol{D}_{x+y} = \boldsymbol{D}_x \boldsymbol{D}_y \quad \Rightarrow \quad \zeta_{x+y} = \zeta_x \zeta_y. \tag{93}$$

Consequently the function ζ given by (17), (19) has necessarily the exponential form

$$\zeta_x = \exp\big(-\Gamma x\big), \quad \text{i.e.} \quad \phi_x = \Gamma x. \tag{94}$$

This form enlightens the comments associated with (23), (25) as it looks much more familiar with the 1-D propagation operator of a harmonic wave: Γ is the complex propagation constant in which the term depending on time is omitted. It is usually expanded as the sum of a real attenuation coefficient and an imaginary wave number,

$$\Gamma = \alpha + \mathrm{j}\frac{2\pi}{\lambda}, \tag{95}$$

($\alpha > 0$, see convention at end of section 2.1.3). Equation (19) now reads

$$\sqrt{Z_x Y_x} = \tanh\big(\Gamma x\big). \tag{96}$$

One introduces the set of new variables $\underline{Z}$ and $\underline{Y}$ with

$$\sqrt{\underline{Z}/\underline{Y}} = Z_c\Big(= \sqrt{Z_x/Y_x}\Big) \quad \text{and} \quad \sqrt{\underline{Z}\underline{Y}} = \Gamma. \tag{97}$$

By combining (92) and (96), the relations between (Z_x, Y_x) and $(\underline{Z}, \underline{Y})$ expand as

$$\begin{cases} Z_x = Z_c \tanh\big(\Gamma x\big) = Z_c \tanh\big(x\sqrt{\underline{Z}\underline{Y}}\big), \\ Y_x = Z_c^{-1} \tanh\big(\Gamma x\big) = Z_c^{-1} \tanh\big(x\sqrt{\underline{Z}\underline{Y}}\big), \end{cases}$$
$$\begin{cases} \underline{Z} = x^{-1} Z_c \Big(\operatorname{arctanh}\sqrt{Z_x Y_x} - \mathrm{j}k\pi\Big), \\ \underline{Y} = x^{-1} Z_c^{-1} \Big(\operatorname{arctanh}\sqrt{Z_x Y_x} - \mathrm{j}k\pi\Big). \end{cases} \tag{98}$$

The ambiguity modulo π of the imaginary parts of $\underline{Z}$ and $\underline{Y}$ occurs because the tanh function is periodic with its imaginary argument (see equation 19). In practical applications where $\underline{Z}$ and $\underline{Y}$ are to be derived from measurements of (Z_x, Y_x), this ambiguity can be solved by using for instance a phase unfolding technique in the frequency domain.

Knowing the parameters $\underline{Z}$ and $\underline{Y}$, the transfer matrix of the two-port corresponding to any length x can be derived by replacing (94) into (20),

$$\boldsymbol{T}_x = \begin{bmatrix} \cosh\big(\Gamma x\big) & Z_c \sinh\big(\Gamma x\big) \\ Z_c^{-1} \sinh\big(\Gamma x\big) & \cosh\big(\Gamma x\big) \end{bmatrix}. \tag{99}$$

Equation (99) is the well known transfer matrix of a transmission line.

Whenever the following condition is met:

$$\big|\Gamma x\big|^2 \ll 1, \tag{100}$$

the first order development of the tanh function in (98) leads to the approximation

$$\begin{cases} Z_x \approx Z_c \Gamma x = \underline{Z} x, \\ Y_x \approx Z_c^{-1} \Gamma x = \underline{Y} x. \end{cases} \tag{101}$$

The terms $\underline{Z}$ and $\underline{Y}$ now clearly appears to be respectively the shunt impedance and the open admittance per unit length. Hence, the transmission media is entirely described by either this pair $(\underline{Z}, \underline{Y})$ of complex values per unit length, or the set made of the characteristic impedance Z_c and the propagation coefficient Γ.

4.2. Evolution with length

It has been seen in the previous sections that the behavior of the system depends largely on the order of magnitude of $|Z_x Y_x|$. It is therefore interesting to study the evolution of this norm with x. Replacing by the notation (95) in (96) leads to

$$\big|Z_x Y_x\big| \approx \frac{\cosh(2\alpha x) - \cos(4\pi x/\lambda)}{\cosh(2\alpha x) + \cos(4\pi x/\lambda)}. \tag{102}$$

For this analysis, it is assumed that $\alpha\lambda \ll 1$. In addition, the domain for the parameter x is limited to $[0, x_{\max} \ll \alpha^{-1}]$. Both conditions are not actually restrictive in most practical situations. Hence, the hyperbolic terms can be replaced by the first order development close to unit. In addition, the argument $2\alpha x$ varies slowly with x compared to the oscillations of the trigonometric terms whose argument is $4\pi x/\lambda$. The numerator and denominator in (102) take simultaneously opposite extreme values at abscissa

$$x_k = k\,\lambda/4 \qquad (k \in \mathbf{N}). \tag{103}$$

The even and odd indices k correspond to peaks of $|Z_x Y_x|^{-1}$ and $|Z_x Y_x|$, respectively. In the vicinity of these extrema, there is the development

$$\left.\begin{aligned} |Z_x Y_x|^{-1}(x_{2m} + \delta x) \\ \text{or} \quad |Z_x Y_x|(x_{2m+1} + \delta x) \end{aligned}\right\}$$
$$\approx (\alpha x)^{-2}\left(1 + \left(\frac{2\pi\delta x}{\alpha\lambda x}\right)^2\right)^{-1}, \tag{104}$$

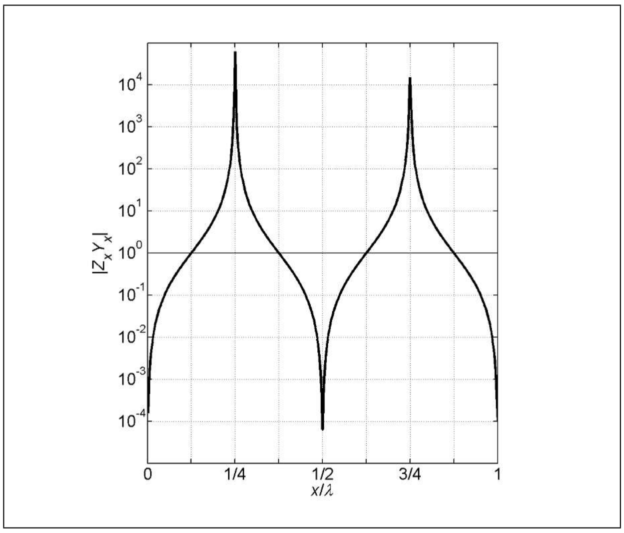

Figure 8. $|Z_xY_x|$ in log scale computed with $\alpha\lambda = 10^{-2}$.

with $|\delta x| \ll \lambda/(2\pi)$.

The order of magnitude of the maximal amplitude, at $\delta x = 0$, is

$$(\alpha x)^{-2} = (k/4)^{-2}(\alpha\lambda)^{-2} \gg 1. \tag{105}$$

From either side of these peaks, the norm decreases by half for a length shift of

$$\Delta x = \alpha\lambda x/(2\pi), \quad \text{i.e.} \quad \frac{\Delta x}{x_k} \approx \frac{\alpha\lambda}{2\pi} \ll 1, \tag{106}$$

which is a very narrow band from the hypothesis. We are more specifically interested in conditions that lead to (31), or conversely (29). The additive unit constant in the second term of (104) can thus be neglected. One finds

$$\begin{gathered} |x - m\lambda/2| < \sqrt{\varepsilon}\lambda/(2\pi) \Rightarrow |Z_xY_x| < \varepsilon, \\ \text{or} \\ |x - (m + 1/2)\lambda/2| < \sqrt{\varepsilon}\lambda/(2\pi) \Rightarrow |Z_xY_x| > \varepsilon^{-1}. \end{gathered} \tag{107}$$

The relative width of these bands is equal to $2\sqrt{\varepsilon}/(m\pi)$. Hence looking at the evolution of $|Z_xY_x|$ with x, there is a regular succession of alternating extrema, the norm being equal to unit in between at abscissa $x_{k+1/2}$ (see Figure 8).

Looking more closely at the minima of $|Z_xY_x|$, it can be noticed that (100) implies

$$\left| Z_xY_x \right| \ll 1. \tag{108}$$

It corresponds to the part of the curve close to the origin in Figure 8, i.e. short lengths,

$$x < \sqrt{\varepsilon}\lambda/(2\pi) \Rightarrow \left| Z_xY_x \right| < \varepsilon. \tag{109}$$

Conversely (108) which holds for the other minima does not imply the linear relation (101). However noteworthy relations can be still derived,

$$\begin{aligned} \left| Z_xY_x \right| &= \left| \tanh^2(\Gamma x - \mathrm{j}k\pi) \right| \ll 1 \\ \Rightarrow &\quad \exists m \in \mathbf{Z}, \left| \Gamma x - \mathrm{j}m\pi \right| \ll 1 \\ \Rightarrow &\quad \begin{cases} Z_x \approx Z_c(\Gamma x - \mathrm{j}m\pi), \\ Y_x \approx Z_c^{-1}(\Gamma x - \mathrm{j}m\pi). \end{cases} \end{aligned} \tag{110}$$

Hence we obtain the following generalization of (101):

$$\begin{aligned} &\left| Z_xY_x \right| \ll 1 \\ &\Rightarrow \quad \exists m \in \mathbf{Z}, \begin{cases} Z_x \approx \underline{Z}x - \mathrm{j}Z_c m\pi, \\ Y_x \approx \underline{Y}x - \mathrm{j}Z_c^{-1} m\pi. \end{cases} \end{aligned} \tag{111}$$

5. Conclusion

The power transfer coefficient through a two-port system has been expressed by means of two non-dimensional parameters: β which is a characteristic parameter of the two-port, and σ that depends on the external impedance load. A typical behavior of the relative power transfer coefficient has been put in evidence when β is not too close to unit (see Figure 6).

A thorough analysis of what is interpreted as a half-wavelength line in most applications has been performed. It accounts also for the case of systems whose length is much smaller than the wavelength. Beyond the trivial result that such a two-port may have then little influence in the power transfer from one port to the other, we exhibited simple and accurate relations between input and output impedances, together with the precise conditions of validity for these approximations (see equation 86). Accordingly, the power transfer coefficient can be close to unit even though the loading impedance is different of the characteristic impedance, but the limits for which this property holds are asserted.

As far as passive, linear, symmetrical two-ports are involved, the results displayed in this paper can give interesting insights whatever the domain of application. Waveguides in acoustics and transmission line in the electrical domain are obvious fields of concern.

References

[1] H. von Helmholtz: Handbuch der physiologischen Optik (Handbook of physiological optics) Leopold Voss, Leipzig, 1867. §16 Satz I, 168-169.

[2] H. von Helmholtz: Über die physikalische Bedeutung des Princips der kleinsten Wirkung (On the physical significance of the principle of least action) Crelle (Journal für die reine und angewandte Mathematik, Berlin) **100** (1886) 137–166 and 213–222.

[3] Lord Rayleigh: The theory of sound. 2nd ed. Dover Publications, New York, 1945 (1st ed. 1877). vol. I, §72 p.93, §77-78 pp97-99, and §107-111 pp150-157.

[4] H. Lamb: On reciprocal theorems in dynamics. Proc. London Math. Soc. **s1-19** (1888) 144–151.

[5] J. L. Lagrange: Analytical mechanics (Mécanique analytique). Mallet-Bachelier, Paris, 1853. T1, 300-302. Revised and annotated by M. J. Bertrand.

[6] H. A. Lorentz: The theorem of Poynting concerning the energy in the electromagnetic field and two general propositions concerning the propagation of light. Amsterdammer Akademie der Wetenschappen **4** (1896) 176.

[7] C.-T. Tai: Complementary reciprocity theorems in electromagnetic theory. IEEE Trans. Antennas Propag. **40** (1992) 675–681.

[8] J. Goedbloed: Reciprocity and EMC measurements. EMC Zurich Symposium, www.ieee.org/organizations/pubs/newsletters/emcs/summer03/jasper.pdf, 2003.

[9] R. J. Potton: Reciprocity in optics. Rep. Prog. Phys. **67** (2004) 717–754.

[10] T. T. Wolde: On the validity and application of reciprocity in acoustical, mechano-acoustical and other dynamical systems. Acustica **28** (1973) 23–23.

[11] B. D. H. Tellegen: The gyrator, a new network element. Philips Res. Rep. **3** (1948) 81–101.

[12] W. E. Kock: The vibrating string considered as an electrical transmission line. J. Acoust. Soc. Am. **8** (1937) 227–233.

[13] T. L. Szabo: Lumped-element transmission-line analog of sound in a viscous medium. J. Acoust. Soc. Am. **45** (1969) 124–130.

[14] G. A. Russell, E. D. Hjerpe: On the analogy between the one-dimensional acoustic waveguide and the electrical transmission line. J. Acoust. Soc. Am. **94** (1993) 583–584.

[15] J. Wu, G. Du: Analogy between the one-dimensional acoustic waveguide and the electrical transmission line for cases with loss. J. Acoust. Soc. Am. **100** (1996) 3973–3975.

[16] V. Easwaran, V. H. Gupta, M. L. Munjal: Relationship between the impedance matrix and the transfer matrix with specific reference to symmetrical, reciprocal, and conservative systems. J. Sound Vib. **161** (1993) 515–526.

[17] J. W. Verheij: A comment on the relationship between reciprocal and symmetrical systems. J. Sound Vib. **170** (1994) 567.

[18] V. Easwaran, V. H. Gupta, M. L. Munjal: Authors'reply. J. Sound Vib. **170** (1994) 568–570.

[19] V. H. Gupta: On independence of reciprocity, symmetry and conservativeness of one-dimensional linear systems. J. Sound Vib. **179** (1995) 547–552.

[20] W. J. Trott: Theory of a passive, reversible, distributed-coupling transducer. J. Acoust. Soc. Am. **34** (1962) 333–337.

[21] W. M. Hall: An acoustic transmission line for impedance measurement. J. Acoust. Soc. Am. **11** (1939) 140–146.

[22] M. L. Kathuriya, M. L. Munjal: Accurate method for the experimental evaluation of the acoustical impedance of a black box. J. Acoust. Soc. Am. **58** (1975) 451–454.

[23] M. L. Kathuriya, M. L. Munjal: Experimental evaluation of the aeroacoustic characteristics of a source of pulsating gas flow. J. Acoust. Soc. Am. **65** (1979) 240–248.

[24] D. H. Keefe: Acoustical wave propagation in cylindrical ducts: Transmission line parameters approximations for isothermal and nonisothermal boundary conditions. J. Acoust. Soc. Am. **75** (1984) 58–62.

[25] S. Scheichl: On the calculation of the transmission line parameters for long tubes using the method of multiple scales. J. Acoust. Soc. Am. **115** (2004) 534–555.

[26] A. J. Rudgers: Equivalent networks for representing the two-dimensional propagation of dilatational and shear waves in infinite elastic plates and in stratified elastic media. J. Acoust. Soc. Am. **91** (1992) 28–38.

[27] H. F. Steenhoek, T. T. Wolde: The reciprocal measurement of mechanical-acoustical transfer functions. Acustica **23** (1970) 301–305.

[28] D. Shmilovitz, I. Yaron, S. Singer: Transmission-line-based gyrator. IEEE Trans. Circuits Systems-I **45** (1998) 428–433.

[29] D. H. Steinbrecher: An interesting impedance matching network (correspondence). IEEE Trans. Microwave Theory and Techniques **15** (1967) 382.