On Passive Symmetrical Two-Ports, Impedance Conversion and Power Transfer

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Summary
The theory of linear passive symmetrical two-ports is reviewed. Many results of practical interest can be derived from the very compact analytical basis of this model. However, such derivations are scattered in the literature. In addition, the demonstrations do not always take advantage of the generality that can be obtained by avoiding implementations built on particular applications. A complete, self consistent analysis is presented here. Attention is focused on power transfer and the relations between the impedances at each port. The specific case of transmission lines is finally addressed.

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1. Introduction
The powerful principles of reciprocity were formulated long ago in the optical and acoustical domains by Von Helmholtz [1, 2] and Lord Rayleigh [3] although Lamb [4] recognized that a generalization of these theorems was already contained in the former Lagrange’s work [5]. Lorentz [6] extended the domain of application to electromagnetism. Extending the scope of the reciprocity principle is always an active field of research (e.g. [7]). Among such works, Goedbloed [8] and Potton [9] give in-depth reviews about reciprocity. The conditions of validity of the reciprocity principle are also a critical issue [10]. When properly defined, most linear passive networks are reciprocal although the theoretical existence of a counter-example has been shown by Tellegen [11] with the gyrator (in the mechanical domain, the actual existence of a passive material that would allow to build such a system is still questioned). The formal analogy of the principle of reciprocity across different domains of physical phenomenon has been pointed out, in particular between electrical and mechanical systems [12, 13, 14, 15]. It justifies the interest of all general properties that can be derived from formal analysis.

Another large class of systems is defined by symmetry. The dependency of symmetry on reciprocity is not a straightforward issue and has been thoroughly addressed [16, 17, 18, 19]. Many physical systems exhibit both the reciprocity and symmetry properties. Hence the related theorems provide powerful tools for dealing with a large range of applications [20], addressing for example impedance measurement [21, 22, 23], modeling of tubes [24, 25], and propagation in layered media [26]. The underlying concern is often power transfer.

This paper addresses the systems that can be described by means of a passive linear symmetrical two-port black box. Most analytical developments are known but scattered in journals and textbooks. The authors think useful to present here a synthetic view of the main results that can be derived from the very few hypotheses that make the basis of this model. In addition, the power transfer coefficients as expressed with non dimensional parameters in section 3, as well as the precise conditions that validate the approximations presented in section 3.5 have not been found elsewhere in the literature. We are primarily interested in the energy transfer through such a system, and in the impedance conversion that its presence involves. The formalism is presented without any explicit reference to propagation equations that would be derived from the physical phenomenon actually involved. The analysis is performed in the context of the harmonic steady state.

The black box that represents the system is passive, which means that it does not contain any internal source. Energy is exchanged with the external world in a way which can be described at each interface by a pair of state variables: a forcing, extensive variable \( V \), and a response, intensive variable \( P \). Their product represents the power fed into or pouring out the system via each terminal. The system is a linear two-port: there are two interfaces with a single pair of scalar conjugate variables at each of them; linear relations link these variables. Hence the scope of this paper does not extend to mechanical sys-
systems that involve generalized forces and displacements because it would require 6 ports per concerned interface [27]. We consider a symmetric system, i.e., the black box remains the same when both ports are permuted. Because of the symmetry, the system is necessarily homogeneous, i.e., both interfaces have the same type (e.g., mechanical-mechanical or electrical-electrical). In the acoustic domain, a large variety of configurations can be taken into account, provided a pair of acoustic pressure and normal velocity defines the conditions at each interface. With an electrical system, \( P \) and \( V \) stand for voltages and currents, respectively (Figure 1). Note that symmetry implies reciprocity when dealing with a linear passive two-port.

For example, the system can be a multilayered symmetric media (solids and/or fluids), whose exterior faces are parallel planes in contact with outside fluids. The definition of the conjugate variables calls here for the time-angular spectra approach (time and spatial Fourier formalism). Considering the invariance of the time and spatial frequencies in the decomposition of the pressure and normal velocity fields at each interface, \( P \) and \( V \) stand for such spectral components at given time and spatial frequencies. In the simplest situation depicted Figure 2, Port 1 and Port 2 are in contact with semi-infinite homogeneous fluids. There is an incident plane wave \( (P_i, V_i) \) at Port 1 that is reflected \( (P_r, V_r) \), and a transmitted wave \( (P_t, V_t) \) out of Port 2. The invariance in the frequencies translates here through the Snell-Descartes relation between \( \theta_i \) and \( \theta_r \). Note that if the fluid is not semi-infinite nor homogeneous, the load condition at Port 2 is modified because the conditions \( (P_2, V_2) \) at the interface will take into account an incoming, reflected wave. In case the direction of the plane waves is normal to the interfaces (1-D propagation), the restriction in the nature of the bounding media is withdrawn. A typical example arises when studying the power transfer between a transducer and the propagating media through a matching layer. Amongst other examples of application in the acoustic domain, the system can be also a waveguide so that the conjugate variables are defined with respect to any given mode.

The matrices that characterize a passive symmetrical two-port are derived in section 2.1. The interpretation in terms of one-way modes, as well as the introduction of the characteristic impedance, are thus straightforward. The relation between the impedances at both interfaces is analyzed in section 2.2. Classical limit cases are pointed out, in association with the propagation terminology (half/quarter/eight wavelength lines). Power transfer is thoroughly addressed in section 3. The analysis is first conducted with respect to the input impedance. The power transfer coefficient exhibits a characteristic pattern in its dependency on the parameters of the two-port. This result does not seem to be known. The optimization of the power transfer within different situations is then discussed. In many particular problems, the parameters that describe the media supporting the system do not depend on the dimension that links the interfaces. Several results displayed in the previous sections are then furthermore expanded in section 4 when this property applies.

2. Passive symmetrical two-port

2.1. Transfer matrix

2.1.1. Two-port

Port 1 and Port 2 are referred as input and output, respectively. Note the convention of sign at the terminals in Figure 1. It implies that power entering the system is counted positive. For an acoustical system, it means that the vectors normal to the interfaces used to define the normal velocities are oriented inward the system at both ports (e.g., see Figure 2). Using this convention, the following impedances are defined according to

\[
Z_1 = P_1 V_1^{-1}, \quad Z_2 = -P_2 V_2^{-1}. \tag{1}
\]

In the acoustical frame, \( Z \) defines a surface impedance because \( V \) refers to the spectral component of a normal velocity. In equation (1), \( Z_2 \) can be interpreted as the impedance that loads the network, whereas \( Z_1 \) stands for this load as seen from the input interface. If \( Z_2 \) is an actual impedance load, then \( \Re\{Z_2\} \geq 0 \). Note that if the exterior media that stands next to Port 2 is an homogeneous, semi-infinite fluid devoid of any source, \( Z_2 \) is simply the characteristic impedance of this media. Furthermore considering a source applied at Port 1, the pressure \( P_1 \) and the velocity \( V_1 \) at this interface cannot be independently imposed, their ratio \( Z_1 \) being dictated by \( Z_2 \) as seen through the blackbox. One chooses to consider the source at the input side, i.e. Port 1, and the radiating media at the output side, i.e. Port 2. The converse configuration could be
indeed considered, i.e. source at Port 2 with surface impedance $Z_1$ at Port 1. Because of the linearity, the general case with sources on both sides can be then handled merely by superposition.

Hence considering that there is no external source at Port 2, let us denote $Z_T$ the shunt impedance and $Y_T$ the open admittance as seen from Port 1, i.e.

$$Z_T = \begin{pmatrix} P_1 V_1^{-1} \end{pmatrix} \bigg|_{P_2=0} = Z_1(Z_2 = 0),$$
$$Y_T = \begin{pmatrix} P_1^{-1} V_2 \end{pmatrix} \bigg|_{V_1=0} = Z_1^{-1}(Z_2^{-1} = 0).$$

(2)

In the electrical transmission line language, it corresponds to measurements on the short or open circuited “stubs”. In the acoustical field, $Z_T$ is the input impedance seen when the output impedance is very weak; the admittance $Y_T$ is the input admittance seen when the output impedance is very large. In both cases, the interface at Port 2 in these idealized experiments is close to a perfect mirror.

The power that is dissipated in a passive two-port cannot be negative, so that the real parts of $Z_T$ and $Y_T$ are positive,

$$\Re \{Z_T\} > 0 \quad \text{and} \quad \Re \{Y_T\} > 0.$$ 

(3)

Note that this property proceeds from a thermodynamic law and is always true. It does not depend on external conditions as it is for example the case with the sign of $\Re \{Z_2\}$ or $\Re \{Z_1\}$. Using (2), the transfer matrix $T$ that describes the two-port reads

$$\begin{bmatrix} P_1 \\ V_1 \end{bmatrix} = T \begin{bmatrix} P_2 \\ -V_2 \end{bmatrix}$$

(4)

with

$$T = \begin{bmatrix} a & b Z_T \\ a Y_T & b \end{bmatrix}. \quad (5)$$

$a$ and $b$ being two complex constants.

Equation (4) can be also reordered in terms of impedance or admittance relations according to

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = Z \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = Y \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}. \quad (6)$$

with

$$Z = Y_T^{-1} \begin{bmatrix} 1 \\ a^{-1} b (1 - Z_T Y_T) \end{bmatrix}$$

and

$$Y = Z_T^{-1} \begin{bmatrix} 1 \\ -a(1 - Z_T Y_T) \end{bmatrix}.$$ 

(7)

2.1.2. Reciprocity

The straightforward definition of reciprocity is the symmetry of the impedance $Z$ (or admittance $Y$) matrix. Reciprocity implies that the determinant of the transfer matrix is equal to unit. In the particular case of a two-port, these properties are even equivalent, as well as the following particular “reciprocity relation”

$$V_1(P_1 = 0, P_2 = P) = V_2(P_1 = P, P_2 = 0).$$

(8)

Hence any of these definitions lead to

$$ab(1 - Z_T Y_T) = 1.$$ 

(9)

A degenerated case arises when $Z_T = Y_T^{-1}$: there is always $Z_1 = Z_T$; both sides of the network are no longer related. There is something inside the black box that hides one port to the other (e.g. perfect reflector). The transfer matrix $T$ is not defined. This case can be depicted as in Figure 3.

For now, we consider the general case $Z_T Y_T \neq 0$.

Because of (9), the transfer matrix that describes a passive reciprocal two-port depends only on three complex parameters $(a, Z_T$ and $Y_T)$,

$$T = \begin{bmatrix} a & a^{-1} Z_T (1 - Z_T Y_T)^{-1} \\ a Y_T & a^{-1} (1 - Z_T Y_T)^{-1} \end{bmatrix} \quad (\Rightarrow ||T|| = 1). \quad (10)$$

Equivalently, the impedance and admittance matrices read

$$Z = Y_T^{-1} \begin{bmatrix} 1 \\ a^{-1} \end{bmatrix} \begin{bmatrix} a^{-1} \frac{1}{a^{-1}} (1 - Z_T Y_T) \end{bmatrix}$$

and

$$Y = Z_T^{-1} \begin{bmatrix} 1 \\ -a(1 - Z_T Y_T) \frac{1}{a^{2}} \end{bmatrix}. \quad (11)$$

2.1.3. Symmetry

For a symmetrical system, it would not matter which port is the input port and which the output port. Hence the impedance $Z$ and admittance $Y$ matrices (7) have equal diagonal and antidiagonal elements: $e_{11} = e_{22}$ and $e_{12} = e_{21}$. Consequently, a symmetrical two-port is also necessarily reciprocal because the matrices $Z$ and $Y$ are then symmetrical. Hence (9) must hold, and the equality of the diagonal terms in (11) implies in addition

$$a^2 = (1 - Z_T Y_T)^{-1}. \quad (12)$$

The transfer matrix (10) is reduced to

$$T = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} Z_T \\ Y_T \end{bmatrix} \quad (13)$$

and the impedance and admittance matrices read

$$Z = Y_T^{-1} \begin{bmatrix} 1 \\ a^{-1} \frac{1}{a^{-1}} \end{bmatrix} \quad \text{and} \quad Y = Z_T^{-1} \begin{bmatrix} 1 \\ -a^{-1} \frac{1}{a^{-1}} \end{bmatrix}. \quad (14)$$

A passive symmetrical two-port is thus characterized by a transfer matrix which depends only on two parameters $(Z_T$, $Y_T)$, whose determinant is unit and diagonal terms are equal. Note that the only knowledge of $Z_T$ and $Y_T$ is not sufficient to select the correct determination of the root of $a^2$ in (12). Additional information is needed to solve this problem: at a very low frequency, it can be for instance...
the sign of $\Re\{P_1 P_2^*\}$ observed when $V_2 = 0$ which dictates the sign of the real part of $a$. This seed being set, complementary techniques such as phase unfolding in the frequency domain can be deployed.

The transfer matrix $T$ can be diagonalized according to $T = U D U^{-1}$ with

$$U = \begin{bmatrix} Z_c & Z_c \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \zeta^{-1} & 0 \\ 0 & \zeta \end{bmatrix},$$

in which appears the characteristic impedance

$$Z_c = \sqrt{Z_T Y_T},$$

and the term $\zeta$

$$\zeta = \left(1 - \frac{Z_T Y_T}{1 + Z_T Y_T}\right)^{1/2}.$$ (15)

These results can be checked in writing

$$T = U D U^{-1} = \frac{1}{2} \begin{bmatrix} \zeta^{-1} + \zeta & (\zeta^{-1} - \zeta) Z_c^{-1} \\ (\zeta^{-1} - \zeta) Z_c^{-1} & \zeta^{-1} + \zeta \end{bmatrix} = a \begin{bmatrix} 1 & Z_T \\ Y_T & 1 \end{bmatrix}.$$ (16)

Thorough the entire paper, the operator $\sqrt{\cdot}$ stands as usual for the root determination whose real part is positive. The ambiguity addressed previously about the coefficient $a$ is now deferred to the root that defines $\zeta$ in (17). This term is written in an exponential form by introducing the complex factor $\phi_T$,

$$\zeta = \exp\left(-\phi_T\right)$$

$$\Leftrightarrow \tanh\left(\phi_T + j k \pi\right) = \sqrt{Z_T Y_T}$$

$$\Leftrightarrow \zeta = (-1)^k \exp\left(-\arctanh\sqrt{Z_T Y_T}\right).$$ (19)

The tanh function being periodic modulo $\pi$ with its imaginary argument, the ambiguity is now made explicit through the integer $k$. With this notation, the transfer matrix reads

$$T = \begin{bmatrix} \cosh \phi_T & Z_c \sinh \phi_T \\ Z_c^{-1} \sinh \phi_T & \cosh \phi_T \end{bmatrix},$$

which comes closer than (13) to the well known transmission line relation.

From the definition of the eigenvectors $[Z_c \pm 1]^T$ which build the matrix $U$ in (15), the output impedances $\pm Z_c$ are left unchanged after the transform $T$,

$$Z_2 = \pm Z_c \Leftrightarrow Z_1 = \pm Z_c$$

(same sign in each equation). These eigenvectors and values can be interpreted in terms of wave propagation as two progressive waves that travel towards opposite directions,

$$T \begin{bmatrix} Z_c \\ 1 \end{bmatrix} = \zeta^{-1} \begin{bmatrix} Z_c \\ 1 \end{bmatrix} \Rightarrow \begin{cases} P_1 \\ V_1 \end{cases} = V \begin{bmatrix} Z_c \\ 1 \end{bmatrix}_{\text{Port}1} \Rightarrow \begin{cases} P_2 \\ -V_2 \end{cases} = \zeta V \begin{bmatrix} Z_c \\ 1 \end{bmatrix}_{\text{Port}2}. \quad (20)$$

$$T \begin{bmatrix} Z_c \\ -1 \end{bmatrix} = \zeta \begin{bmatrix} Z_c \\ -1 \end{bmatrix} \Rightarrow \begin{cases} P_1 \\ V_1 \end{cases} = \zeta V \begin{bmatrix} Z_c \\ -1 \end{bmatrix}_{\text{Port}1} \leadsto \begin{cases} P_2 \\ -V_2 \end{cases} = V \begin{bmatrix} Z_c \\ -1 \end{bmatrix}_{\text{Port}2}.$$ (21)

In other words, any condition at the output can be split into two projections upon these eigenvectors,

$$\begin{bmatrix} P_2 \\ -V_2 \end{bmatrix} = \frac{P_1}{Z_c (1 + \chi_2)} \left(\begin{bmatrix} Z_c \\ 1 \end{bmatrix} + \chi_2 \begin{bmatrix} Z_c \\ -1 \end{bmatrix}\right). \quad (22)$$

with $\chi_2 = \frac{Z_2 - Z_c}{Z_1 + Z_c}$. (23)

which gives back to the input side:

$$\begin{bmatrix} P_1 \\ V_1 \end{bmatrix} = \frac{P_2}{Z_c (1 + \chi_2)} \left(\begin{bmatrix} Z_c \\ 1 \end{bmatrix} + \chi_2 \begin{bmatrix} Z_c \\ -1 \end{bmatrix}\right). \quad (24)$$

The eigenvalue $\zeta$ (17) is the propagation operator that transforms a wave traveling from one interface to the other, whereas the inverse value $\zeta^{-1}$ in (25) accounts for retro-propagation, i.e. finding back the wave at the interface from where it originated. Considering that there is a source only on the Port 1 side, the term $\chi_2$ in (24) can be interpreted as the reflection coefficient observed at Port 2. Equivalently the ratio between the forward and backward waves at Port 1 is derived from (25),

$$\chi_1 = \chi_2 \Rightarrow \frac{Z_1 - Z_c}{Z_1 + Z_c} = \frac{Z_1}{Z_2}.$$ (25)

The magnitudes $|\chi_1|$ and $|\chi_2|$ differ on both sides only because of the losses encountered in the black box, which are taken into account by the real part of $\phi_T$. Hence because of the convention stated above and the notation given in (19), that real part is always positive.

2.2. Characteristic impedance, input and output impedances

Looking now at the relation between $Z_1$ and $Z_2$ through the transfer matrix (13), there is

$$Z_1 = \frac{Z_2 + Z_T}{1 + Z_2 Y_T}, \quad Z_2 = \frac{Z_1 - Z_T}{1 - Z_1 Y_T}.$$ (26)

It is easy to derive again directly from (27) that the characteristic impedance $Z_c$ is the remarkable load value at the output side that is seen unchanged at the input side. Whenever $Z_1$ and $Z_2$ depart from this value, (27) does not give a convenient formula that links explicitly the differences $Z_2 - Z_c$ and $Z_1 - Z_c$. Nonetheless there is

$$Z_1 Z_2 = Z_c^2 \left(1 + \frac{Z_c - Z_1}{Z_c \sqrt{Z_T Y_T}}\right).$$ (27)
(28) allows to point out three particular cases that depend on the order of magnitude of \( \sqrt{Z_T Y_T} \). With the help of (17), (19), there is

**Case 1**

\[
|\sqrt{Z_T Y_T}| \ll 1 \quad \Rightarrow \quad Z_1 \approx Z_2 \quad \text{(29)}
\]

\[
\Rightarrow \quad \phi_T = jk \pi \quad \Rightarrow \quad T = (-1)^k I \quad (k \in \mathbb{Z}) \quad \text{(30)}
\]

in which \( I \) denotes the 2×2 identity matrix. Because \( |\arg(\zeta)| = 0 \mod \pi \), this case is related to halfwavelength lines. The input impedance is the same as the output impedance, although not equal to the characteristic impedance \( Z_c \). It implies important practical consequences. This case is thoroughly investigated in section 3.5.

**Case 2**

\[
|\sqrt{Z_T Y_T}| \gg 1 \quad \Rightarrow \quad \left\{ \begin{array}{l}
Z_1 Z_2 \approx Z_2^c \\
z^2 \approx -1
\end{array} \right. \quad \text{(31)}
\]

\[
\Rightarrow \quad \phi_T = j\pi \left( \frac{1}{2} + k \right) \quad \text{(32)}
\]

\[
\Rightarrow \quad T = (-1)^k j \begin{bmatrix}
0 & Z_c \\
Z_c^{-1} & 0
\end{bmatrix} \quad (k \in \mathbb{Z})
\]

The propagation from one port to the other involves a phase rotation of a quarter of a circle (\( \arg(\zeta) = \pi/2 \mod \pi \)), which is the signature of the so-called quarterwavelength lines. The matrix \( T \) can be also interpreted as the transfer matrix of a gyrotary with gyration impedance \( Z_c \) connected in cascade with a \( \pm \pi/2 \) phase shifting network [28]: it transfers a weak impedance into a very large impedance and vice versa. Less drastically, a \( \lambda/4 \) system can be used to perform a conversion between different impedances. For instance in the acoustic domain, a \( \lambda/4 \) layer with impedance \( Z_c \) is used to match the low impedance \( Z_1 \) to a propagating medium with the high impedance \( Z_1 \approx Z_2^c Z_2^{-1} \) of a transducer ceramic.

**Case 3**

Whenever the characteristic impedance \( Z_c \) is real, a remarkable property arises at the transition point for which the norm \( |Z_T Y_T| \) is exactly equal to unit

\[
|Z_T Y_T| = 1 \quad \text{and} \quad Z_c \in \mathbb{R} \quad \text{(33)}
\]

\[
\Rightarrow \quad \left\{ \begin{array}{l}
\forall Z_2 \in \mathbb{R}, |Z_2| = Z_c, \\
\text{if } |Z_2| = Z_c, \forall \arg(Z_2), \quad Z_1 \in \mathbb{R}
\end{array} \right.
\]

Whatever is the value of a purely resistive load, the norm of the input impedance is equal to the norm of the characteristic impedance. Conversely, whenever the norm of the load is equal to the norm of the characteristic impedance, the input impedance is real. The propagation coefficient \( \phi_T \) and the transfer matrix can be derived without assuming that \( Z_c \) is real:

\[
\sqrt{Z_T Y_T} = \exp(j\theta) \quad \Rightarrow \quad \zeta = -j\tan(\theta/2) \quad \text{(34)}
\]

\[
\Rightarrow \quad \phi_T = -\frac{1}{2} \ln \left( \tan \left( \frac{\theta}{2} \right) \right) + j \left( \operatorname{sgn}(\theta) \frac{\pi}{4} + k\pi \right).
\]

\[
\Rightarrow \quad T = \frac{1}{2} (-1)^k \sqrt{\frac{j}{\sin \theta}} \begin{bmatrix}
\exp(-j\theta/2) & Z_c \exp(j\theta/2) \\
Z_c^{-1} \exp(j\theta/2) & \exp(-j\theta/2)
\end{bmatrix}.
\quad (k \in \mathbb{Z}). \quad \text{Note: } \theta \neq 0 \text{ because } Z_T \neq Y_T^{-1} \text{ as stated in section 2.1.2.}
\]

This situation is known [29] as a \( \lambda/8 \) impedance transformer (\( |\arg(\zeta)| = |\Im{\phi_T}| = \pm \pi/4 \mod \pi \)).

### 3. Power transfer

#### 3.1. Notations

We study here the power transfer from a source at the input of a passive linear symmetrical two-port, whose transfer matrix is given in (13), towards an impedance \( Z_2 \) that loads the output. The definition and meaning of \( Z_2 \) has been already discussed at the beginning of section 2. With electrical systems or acoustical waveguides, the source supplies energy per time unit. With systems that involve plane interfaces (and therefore angular spectra), the source delivers energy per time unit and per surface unit. In the following, we use the terminology “power” for both meanings.

The source delivers the power \( W_1 \) through the input port

\[
W_1 = \frac{1}{2} \Re \{ P_1 V_1^* \} = \frac{1}{2} |P_1||V_1|\cos \theta_1 \quad \text{(36)}
\]

\[
\text{with } \theta_1 = \arg(Z_1).
\]

(\( X^* \) denotes the complex conjugate of \( X \)).

Let us recall that with acoustic systems, \( (P_1, V_1) \) refers to the conditions at the input interface. Hence in the example given with Figure 2, \( W_1 \) should not be confused with the power associated with the incoming incident wave \( (P_1, V_1) \).

The power \( W_2 \) that is dissipated in the load \( Z_2 \) reads

\[
W_2 = \frac{|1 + Z_T Y_T^*||P_1||V_1|\cos \delta_1 - \Re \{ Y_T^* \}|P_1|^2 - \Re \{ Z_T \}|V_1|^2}{2|1 - Z_T Y_T^*|} \quad \text{(37)}
\]

with \( \delta_1 = \theta_1 - \arg(1 + Z_T Y_T^*) \). \quad (38)

The power delivered by the source and the power dissipated by the load are related through

\[
W_2 = \frac{\mu}{2} |P_1||V_1| = \eta W_1 \quad \text{(39)}
\]

\[
\text{with } \mu = \eta \cos \theta_1 \leq \eta < 1.
\]

The coefficient \( \eta \) gives the relative amount of power that is actually transferred from the source to the load: the two-port dissipates the ratio \( 1 - \eta \) of the total power delivered by the source. The coefficient \( \mu \) characterizes the fraction of power that is dissipated in the load compared to the maximal power \( |P_1||V_1|/2 \) that the source can potentially deliver with the amplitudes \( |P_1| \) and \( |V_1| \):

\[
\mu = \frac{|1 - Z_T Y_T^*|\Re \{ Z_2 \}}{|1 + Z_T Y_T^*||Z_2 + Z_T|} \quad \text{(40)}
\]

\[
= \frac{\Re \{(Z_1 - Z_T)(1 - Z^*_T Y_T^*)\}}{|1 - Z_T Y_T^*||Z_1|}.
\]
3.2. Optimal input impedance
3.2.1. Optimal phase
Given the amplitudes $|P_1|$ and $|V_1|$ at the input side, one seeks first at the phase $\theta_1$ of the impedance $Z_1$ that maximizes the power $W_2$ dissipated in $Z_2$. Hence $\mu$ is maximal when $\delta_1 = 0$ (38), i.e.

$$\theta_{\text{opt}} = \arg \{1 + Z_T Y^*_T\}$$

(41)

i.e.

$$Z_1 = |Z_1| \frac{1 + Z_T Y^*_T}{|1 + Z_T Y^*_T|}.$$  

When this condition on the phase is true, the power that is dissipated in the load $Z_2$ reads

$$W_2 = \frac{(R_{\min} + R_{\max})|P_1||V_1| - |P_1|^2 - R_{\min}R_{\max}|V_1|^2}{2(R_{\min} - R_{\max})}$$

$$= \frac{R_{\max}|P_1|v}{R_{\min} - R_{\max}}$$

(42)

with

$$|p| = |P_1| - R_{\min}|V_1|, \quad |v| = |V_1| - R_{\max}^{-1}|P_1|$$

and

$$R_{\min} = \frac{2Re\{Z_T\}}{|1 + Z_T Y^*_T| + |1 - Z_T Y^*_T|},$$

$$R_{\max}^{-1} = \frac{2Re\{Y_T\}}{|1 + Z_T Y^*_T| + |1 - Z_T Y^*_T|}.$$  

It can be checked that there is always $0 < R_{\min} < R_{\max}$. The locus of the solutions $\{|P_1|, |V_1|\}$ that lead to dissipate at best the given power $W_2$ in the load is the hyperbola with axes $|p| = 0$ and $|v| = 0$ (see Figure 4). All the physical solutions are necessarily located in the sector that is bounded by these two half lines: the power that is dissipated in the load cannot be negative. In other words, the real part of the load $Z_2$ being positive, the norm $|Z_1|$ is in the range

$$R_{\min} < |Z_1| < R_{\max}.$$  

(44)

Let us introduce the geometrical mean $R_m$ of the resistances $R_{\min}$ and $R_{\max}$.

$$R_m = \sqrt{R_{\min}R_{\max}} = \sqrt{\frac{Re\{Z_T\}}{Re\{Y_T\}}}.$$  

(45)

Notice that $R_m$ is not equal to $Z_m$. In order to browse the domain of variation of the norm $|Z_1|$ (44), one uses the scalar parameter $\sigma$ that is defined in the domain $[0, 1]$ so that $|Z_1|(|\sigma = 0) = R_{\min}$ and $|Z_1|(|\sigma = 1) = R_{\max}$:

$$|Z_1|(|\sigma) = R_{\min}^{1-\sigma}R_{\max}^{\sigma} = R_m\left(\frac{R_{\min}}{R_{\max}}\right)^{0.5-\sigma}.$$  

(46)

with $0 < \sigma < 1$.  

When the phase of $Z_1$ is optimal, the transfer coefficient $\mu$ is equal to

$$\mu_{\delta_1=0}(|Z_1|) = \frac{(R_{\max} - |Z_1|)(|Z_1| - R_{\min})}{|Z_1||R_{\max} - R_{\min}|}$$

$$= \frac{1}{\beta} \left(1 - \sqrt{1 - \beta^2} \cosh \left((\sigma - 0.5) \ln \frac{1 - \beta}{1 + \beta} \right)\right),$$

in which is introduced the coefficient $\beta$ that characterizes the relative difference between the resistances $R_{\min}$ and $R_{\max}$.

$$0 < \beta = \frac{R_{\max} - R_{\min}}{R_{\max} + R_{\min}} = \frac{1 - Z_T Y_T}{1 + Z_T Y_T} < 1,$$

$$\frac{R_{\min}}{R_{\max}} = \frac{1 - \beta}{1 + \beta}.$$  

(48)

Whenever the orders of magnitude (29) or (31) hold, then the coefficient $\beta$ tends towards unity, which is equivalent to $R_{\min} \ll R_{\max}$.

Figure 5 shows the mapping of $\mu_{\delta_1=0}(\beta, \sigma)$. One observes the following properties

$$\mu_{\delta_1=0} = 0, \quad \mu_{\delta_1=0} = 0,$$

$$\beta = 1.$$  

(49)

The latter approximation close to the limit $\beta = 1$ should not hide the sharp evolution of $\mu(\beta)$, as well as the sharp variation of $\mu(\sigma)$ in the vicinity of the boundaries $\sigma = 0$ and $\sigma = 1$,

$$\frac{\partial \mu_{\delta_1=0}}{\partial \beta} \bigg|_{0.5-\sigma-0.5} = + \infty.$$  

(50)

From (47), the power transfer is maximal when

$$\sigma = 1/2, \quad \text{i.e.} \quad |Z_1|_{\text{opt}} = R_m.$$  

(51)
Given the dissipated power $W_2$, the power that is required at the source is thus minimal when the load $Z_2$ is equal to

$$Z_{2\text{opt}} = \frac{Z_{1\text{opt}} - Z_T}{1 - Z_{1\text{opt}} Y_T} = \frac{\Re \{ Z_T \} - j 3 m \{ Z_{1\text{opt}} \}}{\Re \{ Y_T Z_{1\text{opt}} \}}$$

$$= \sqrt{\Re \{ Z_T \} \Re \{ Y_T \} - j \sin \beta_{1\text{opt}}}$$

with

$$Z_{1\text{opt}} = R_m \exp(j \theta_{1\text{opt}}).$$

The coefficient $\mu$ is then equal to (see Figure 5b)

$$\mu_{\text{opt}} = \mu_{\text{opt}}(\beta, \sigma) = \frac{Z_{2\text{opt}}}{Z_{1\text{opt}}} = \frac{\beta}{1 + \sqrt{1 - \beta^2}} \sigma = 1/2$$

$$\frac{\beta}{2} < \mu_{\text{opt}} = \frac{\beta}{1 + \sqrt{1 - \beta^2}} < \beta \leq 1.$$
written in terms of the previously defined parameters \( \delta_1 \) (38), \( R_{\text{min}} \) and \( R_{\text{max}} \) (43), and \( \beta \) (48),

\[
W_2 = \left( \frac{R_{\text{min}} + R_{\text{max}}}{|P_1| |V_1| \cos \delta_1 - |P_1|^2 - R_{\text{min}} R_{\text{max}} |V_1|^2} \right) ^{2} \frac{r_{\text{max}} - r_{\text{min}}}{2 |p'| |v'|} \left( \frac{r_{\text{max}} - r_{\text{min}}}{R_{\text{max}} - R_{\text{min}}} \right) \]

\[
= \frac{r_{\text{max}}}{R_{\text{max}} - R_{\text{min}}} \frac{|p'| |v'|}{2}.
\]

with

\[
|p'| = |P_1| - r_{\text{min}} |V_1|, \quad |v'| = |V_1| - r_{\text{max}}^{-1} |P_1|
\]

and

\[
r_{\text{min}} = \frac{1 + \beta}{\cos \delta_1 + \sqrt{\beta^2 - \sin^2 \delta_1}} R_{\text{min}},
\]

\[
r_{\text{max}}^{-1} = \frac{1 + \beta}{\cos \delta_1 + \sqrt{\beta^2 - \sin^2 \delta_1}} R_{\text{max}}^{-1}.
\]

The geometrical mean of \( r_{\text{min}} \) and \( r_{\text{max}} \) is the same as when the phase is optimal,

\[
\sqrt{r_{\text{min}} r_{\text{max}}} = \sqrt{R_{\text{min}} R_{\text{max}}} = R_m.
\]

The phase \( \theta_1 \) of \( Z_1 \), and consequently the difference \( \delta_1 \), are constrained by the impedance \( Z_2 \), which translates into the inequality

\[
| \sin \delta_1 | \leq \beta \leq 1.
\]

Being given the phase difference \( \delta_1 \), the locus of the solutions \((|P_1|, |V_1|)\) that lead to dissipate at best a given power \( W_2 \) in the load is the hyperbola whose axes are \( |p'| = 0 \) and \( |v'| = 0 \). The physical solutions lay in a sector bounded by two lines whose angle is now smaller than if the phase is optimal,

\[
R_{\text{min}} \leq r_{\text{min}} \leq R_m \leq r_{\text{max}} \leq R_{\text{max}}
\]

and

\[
0 \leq \frac{r_{\text{max}} - r_{\text{min}}}{R_{\text{max}} - R_{\text{min}}} = \sqrt{1 - \beta^2 \sin^2 \delta_1} \leq 1.
\]

The domain \([r_{\text{min}} r_{\text{max}}]\) of the norm \(|Z_1|\) can be browsed again with the parameter \( \sigma \) which is now defined so that \(|Z_1| (\sigma = 0) = r_{\text{min}} \) and \(|Z_1| (\sigma = 1) = r_{\text{max}} \).

\[
|Z_1| (\sigma, \delta_1) = r_{\text{min}}^{1-\sigma} r_{\text{max}}^{\sigma} = R_m \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right)^{0.5-\sigma}.
\]

with \( 0 < \sigma < 1 \).

With these notations, the general expression of the transfer coefficient \( \mu \) (40) reads

\[
\mu = \frac{(r_{\text{max}} - |Z_1|)(|Z_1| - r_{\text{min}})}{|Z_1| (R_{\text{max}} - R_{\text{min}})} = \frac{1}{\beta} \left( \cos \delta_1 - \sqrt{1 - \beta^2} \cosh \left( (\sigma - 0.5) \ln \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right) \right) \right).
\]

where

\[
\frac{r_{\text{min}}}{R_{\text{max}}} = \frac{\cos \delta_1 - \sqrt{\beta^2 - \sin^2 \delta_1}}{\cos \delta_1 + \sqrt{\beta^2 - \sin^2 \delta_1}}
\]

This result can be compared with (47), in which in particular \( R_{\text{min}}/R_{\text{max}} \) is given by (48). Given a constant module \(|Z_1|\), the loss of efficiency because of the non optimal phase (\( \delta_1 \neq 0 \)) can be quantified with

\[
\mu_{\delta_1 \neq 0} (|Z_1|) = \mu_{\delta_1 = 0} (|Z_1|) - 2 \beta \sin^2 (\delta_1/2).
\]

recalling that this latter expression is only meaningful as long as \( r_{\text{min}} < |Z_1| < r_{\text{max}} \).

When the phase is not optimal, the maximal transfer is still obtained with the same norm \(|Z_{\text{opt}}| = R_m \) as when the phase is optimal (see equation (51)). The transfer coefficient is

\[
\mu_{\delta_1} (|Z_{\text{opt}}|) = \beta^{-1} \left( \cos \delta_1 - \sqrt{1 - \beta^2} \right).
\]

to be compared with (54).

### 3.3. Optimal source at constant pressure or constant velocity

The previous developments are founded on the search for maximal transfer coefficients without any constraints on the amplitudes \(|P_1|\) and \(|V_1|\) at the input interface, but for their ratio. This section addresses the conditions that maximize the amount of power dissipated in the load, being given a constant amplitude \(|V_1|\) or \(|P_1|\) at the source. It comes back to look at the hyperbola that are tangent to the lines \( V = |V_1| \) and \( P = |P_1| \). The condition (41) about the phase of the input impedance is still assumed to be fulfilled (\( \delta_1 = 0 \)).

Given a constant amplitude \(|V_1|\), the configuration that leads to the maximal power dissipation in the load is derived by solving \( dW_2/dP_1 = 0 \). From (27), (42), there is

\[
|Z_1| = \frac{1}{2} (R_{\text{min}} + R_{\text{max}}) \Rightarrow Z_1 = 1 + Z_2 Y^*_T \Rightarrow Z_2 = Y^*_T^{-1}.
\]

Accordingly given a constant amplitude \(|P_1|\), the input impedance that leads to dissipate the maximal power in the load is obtained by solving \( dW_2/dV_1 = 0 \). From (27), (42), there is

\[
|Z_1|^{-1} = \frac{1}{2} (R_{\text{min}}^{-1} + R_{\text{max}}^{-1}) \Rightarrow Z_1 = \frac{2 \text{Re}[Z_T]}{1 + Z_T Y_T} \Rightarrow Z_2 = Z_T^*.
\]

In both cases, the transfer coefficient \( \mu_{\delta_1 = 0} \) is reduced to

\[
\mu_0 = \beta/2 = (1 + \sqrt{1 - \beta^2}) \mu_{\text{opt}}/2 \geq \mu_{\text{opt}}/2.
\]
Hence $\mu_0$ is larger than half the optimal transfer coefficient. The solutions (69) and (70) correspond to the following locus of the parameters $(\sigma, \beta)$,

$$
\sigma^{-1} = \begin{cases} 
1 - \frac{\ln(1 + \beta)}{\ln(1 - \beta)} & \text{at const. } |V_i|, \\
1 - \frac{\ln(1 - \beta)}{\ln(1 + \beta)} & \text{at const. } |P_i|. 
\end{cases}
$$

(72)

The curves that represent equations (71), (72) are drawn in red and blue in Figures 5–6.

When the orders of magnitude (31) or (29) apply, $\beta \approx 1$ so that $\mu_0 \approx 1/2$.

3.4. Input impedance equal to load

The equality of the impedances $Z_1$ and $Z_2$ with the characteristic impedance $Z_c$ (21) does not satisfy a priori the criteria that optimize the power transfer. From (39), (40), the transfer coefficients read in that case

$$
Z_1 = Z_2 = Z_c \Rightarrow
\begin{cases}
\mu_c = \eta_c \cos(\arg(Z_c)), \\
\eta_c = \left| \frac{1 - \sqrt{Z_1 Y_T}}{1 + \sqrt{Z_1 Y_T}} \right| = \zeta^2 \zeta = \exp(-2\Re(\phi_T)) < 1.
\end{cases}
$$

(73)

When the orders of magnitude (31) or (29) apply, there is no significant power dissipation in the two-port,

$$
\begin{cases}
|\sqrt{Z_1 Y_T}| \gg 1 \\
|\sqrt{Z_1 Y_T}| \ll 1
\end{cases}
$$

and $Z_{12} = Z_c \Rightarrow \eta_c = 1.$

(74)

At the transition configuration (34), the transfer coefficient $\eta$ reads

$$
|\sqrt{Z_1 Y_T}| = 1 \text{ and } Z_{12} = Z_c \Rightarrow \eta_c = \left| \tan \left( \frac{1}{2} \arg \left( \sqrt{Z_1 Y_Z} \right) \right) \right|.
$$

(75)

In that case, there is no loss in the network if

$$
\arg \left( \sqrt{Z_1 Y_T} \right) \approx \pi/2.
$$

which can be fulfilled only if $Z_T$ and $Y_T$ are both pure imaginary parameters (equation (3) and $Z_T Y_T \in \mathbb{R}^+$).

According to its definition, the characteristic impedance $Z_c$ is real only if the phases of $Z_T$ and $Y_T$ are equal. When this condition is met, the optimal phase of $Z_1$ (41) is null ($Z_T Y_T^* \in \mathbb{R}$), and $Z_2 = Z_1 = Z_c$ is the optimal solution given by (45), (52), (53),

$$
\arg \{Z_T\} = \arg \{Y_T\} \Rightarrow \arg \{Z_{1\text{opt}}\} = \arg \{Z_{2\text{opt}}\} = \arg \{Z_c\} \in \mathbb{R}^+.
$$

(76)

The expression of $\eta_c$ is still given by (73), but $Z_c$ is now real so that

$$
\mu_c = \mu_{\text{opt}} = \eta_c = \eta_{\text{opt}}.
$$

(77)

The efficiency of the source and the power transfer are close to perfect ($\mu = \eta = 1$) whenever (74), or (75) together with $Z_T Y_T \in \mathbb{R}^+$, applies.

3.5. Particular case $|Z_T| \ll |Y_T^{-1}|$

Several additional results can be derived in the particular case of practical interest defined by the following order of magnitude:

$$
|Z_T Y_T| \ll 1.
$$

(78)

Recalling that the real parts of $Z_T$ and $Y_T$ are both positive, it must be kept in mind in the derivations of this section that (3), (78) are equivalent to

$$
0 < \frac{\Re\{Z_T\}}{\Im\{Z_T\}} \leq |Z_T| \leq |Y_T^{-1}| \leq \left\{ \frac{\Re\{Y_T\}}{\Im\{Y_T\}} \right\}.
$$

(79)

It implies straightforward approximations on parameters in section 3.2,

$$
\theta_1 \approx \delta_1, \quad \beta \rightarrow 1, \quad \left\{ \frac{\Re_{\min}}{\Im_{\min}} \approx \Re\{Z_T\} \right\}
$$

$$
\text{and } \left\{ \frac{r_{\min}}{r_{\max}} \approx \Re_{\max}/\cos \theta_1, \quad \frac{r_{\min}^1}{r_{\max}^1} \approx \Re_{\max}/\cos \theta_1. \right\}
$$

(80)

An important consequence is that the phase of $Z_1$ is optimal when being null. Also from the definition (16), the order of magnitude of the characteristic impedance $Z_c$ is such that

$$
|Z_T| \ll |Z_c| \ll |Y_T^{-1}|.
$$

(81)

On the other hand, it has been already pointed out that $\beta$ being close to unit does not necessarily imply obvious simplifications on the transfer functions.

The norms $|Z_1|$ or $|Z_2|$ cannot have simultaneously the same order of magnitude as $|Y_T|^{-1}$ and $|Z_T|$. In addition, (27) shows that $|Z_1|$ and $|Z_2|$ share the same order of magnitude relatively to $|Y_T|^{-1}$ and $|Z_T|$. In other words, one of the following non exclusive propositions is necessarily true, and depending on the case, the differences between impedances or between admittances, as well as the transfer coefficient $\mu$ (40) read

$$
\begin{cases}
( |Z_1| \ll |Y_T^{-1}|, \quad |Z_2| \ll |Y_T^{-1}| ) \\
\Rightarrow |Z_1 - Z_2| \approx |Z_T|, \quad |Z_{1\text{opt}} - Z_{2\text{opt}}| = |Z_c| \in \mathbb{R}^+. 
\end{cases}
$$

(82)

or

$$
\begin{cases}
( |Z_T| \ll |Z_1|, \quad |Z_T| \ll |Z_2| ) \\
\Rightarrow |Z_T^{-1} - Z_1^{-1}| \approx |Y_T^{-1}|, \quad |Z_{1\text{opt}}^{-1} - Z_{2\text{opt}}^{-1}| = |Z_c| \in \mathbb{R}^+. 
\end{cases}
$$

In any instance, provided the impedance $Z_1$ is not essentially reactive ($\tan \theta_1 \gg 1$), the fraction of power that is dissipated in the network is given by

$$
1 - \eta = \Re\{Z_1\} \Re\{Z_T\} + \Re\{Z_1^{-1}\} \Re\{Y_T\}.
$$

(83)
The results stated in (82), (83) lead to interesting simplifications if additional conditions are introduced. Hence, when both orders of magnitude given in (82) are simultaneously met, the load as seen through the network is not significantly modified, although the common value can depart from the characteristic impedance $Z_c$. That is the result already pointed out in (29). In addition, still provided the impedance $Z_{1,2}$ is not essentially reactive, the coefficient $\mu$ can be directly estimated from the argument $\theta_1$ of the load impedance (because $\theta_1 = \theta_2$), and the power that is dissipated in the network is negligible,

$$\left| Z_T \right| \ll \left| Z_{1,2} \right| \ll \left| Y_T \right|^{-1}$$

(84)

and

$$\Rightarrow \begin{cases} Z_1 = Z_2, \\ \mu = \cos \theta_{1.2}, \\ \eta = 1. \end{cases}$$

In order to complete the discussion, it can be checked that when $|\theta_{1.2}|$ is close to $\pi/2$, there is still $Z_1 = Z_2$, the transfer coefficient $\mu$ tends toward zero, but $\eta$ is not necessarily equal to unit.

Starting from (82), (83), another condition of great practical importance that is less restrictive than (84) and that still leads to a negligible relative amount of power dissipated in the network can be also derived,

$$\Re \left\{ Z_T \right\} \ll \left| Z_{1,2} \right| \ll \Re^{-1} \left\{ Y_T \right\}$$

and

$$\tan \theta_{1.2} \neq 1$$

(85)

$$\Rightarrow \begin{cases} \mu = \cos \theta_1, \\ \eta = 1. \end{cases}$$

Compared to (84), the load $Z_2$ and the impedance $Z_1$ seen at the input are not necessarily equal, but the possible significant differences in the impedances $Z_{1,2}$ or admittances $Z_{1,2}^*$ are an imaginary component

$$\Re \left\{ Z_T \right\} \ll \left| Z_{1,2} \right| \ll \Re^{-1} \left\{ Y_T \right\}$$

$$\tan \theta_{1.2} \neq 1$$

and

$$\Rightarrow \begin{cases} Z_1 = Z_2 \approx j \Re \{ Z_T \} \\ \mu = \eta \approx 1. \end{cases}$$

The only conditions (78) and (85) leave a large range in the choice of $Z_1$, which thus can differ significantly from the characteristic impedance $Z_c$, still avoiding power losses in the network. Nevertheless, the source does not necessarily work in an optimal regime because it provides only the ratio $\cos \theta_1$ of the total power that could be potentially delivered. For the source to work optimally, the input impedance $Z_1$ must be purely resistive,

$$\Re \left\{ Z_T \right\} \ll Z_1 = R \ll \Re^{-1} \left\{ Y_T \right\}$$

$$\Rightarrow \mu = \eta \approx 1.$$
studied phenomenon do not depend on the x parameter, whereas a single pair of intensive and extensive variables \((P_i, V_i)\) can be associated to each value of x. A classical example is the modeling of an electrical cable: the transfer matrix associated with a piece of the line is equal to the product of the matrices associated with all parts that build any partition of the initial piece. In the acoustical domain, other trivial examples are obtained when studying propagation between two planes (with distance x) in a homogeneous media or in a tube with constant cross section. The model is also valid whatever the dispersive character of the propagation is (e.g in waveguides).

Let us denote \(Z_x = Z_T\) and \(Y_x = Y_T\) the parameters associated with the matrix \(T_x\). The commutativity of (91) implies that the ratio \(Z_x/Y_x\) does not depend on the length x.

\[
\forall x, \quad Z_x/Y_x = Z'_c. \tag{92}
\]

That is why (16) can be said to define the characteristic impedance of a media when (91) holds.

In the decomposition (15), there is only the diagonal matrix \(D\) that depends on the length x. Hence the property (91) incorporates into this matrix according to

\[
\forall x, y \in \mathbb{R} \quad D_{x+y} = D_x D_y \Rightarrow \xi_{x+y} = \xi_x \xi_y. \tag{93}
\]

Consequently, the function \(\xi\) given by (17), (19) has necessarily the exponential form

\[
\xi_x = \exp(-\Gamma x), \quad \text{i.e.} \quad \phi_x = \Gamma x. \tag{94}
\]

This form enlightens the comments associated with (23), (25) as it looks much more familiar with the 1-D propagation operator of a harmonic wave: \(\Gamma\) is the complex propagation constant in which the term depending on time is omitted. It is usually expanded as the sum of a real attenuation coefficient and an imaginary wave number,

\[
\Gamma = \alpha + j \frac{2\pi}{\lambda}. \tag{95}
\]

\((\alpha > 0, \text{see convention at end of section 2.1.3})\). Equation (19) now reads

\[
\sqrt{Z_x Y_x} = \tanh(\Gamma x). \tag{96}
\]

One introduces the set of new variables \(Z\) and \(Y\) with

\[
\sqrt{Z/Y} = Z_c = \begin{cases} \sqrt{Z_x/Y_x} \text{ and } \sqrt{Z Y} = \Gamma. \end{cases} \tag{97}
\]

By combining (92) and (96), the relations between \((Z_x, Y_x)\) and \((Z, Y)\) expand as

\[
\begin{cases}
Z_x = Z_c \tanh(\Gamma x) = Z_c \tanh(x \sqrt{Z Y}), \\
Y_x = Z_c^{-1} \tanh(\Gamma x) = Z_c^{-1} \tanh(x \sqrt{Z Y}), \\
Z = x^{-1} Z_c \left(\arctanh \sqrt{Z_x Y_x} - j k \pi\right), \\
Y = x^{-1} Z_c^{-1} \left(\arctanh \sqrt{Z_x Y_x} - j k \pi\right).
\end{cases} \tag{98}
\]

The ambiguity modulo \(2\pi\) of the imaginary parts of \(Z\) and \(Y\) occurs because the tanh function is periodic with its imaginary argument (see equation 19). In practical applications where \(Z\) and \(Y\) are to be derived from measurements of \((Z_x, Y_x)\), this ambiguity can be solved by using for instance a phase unfolding technique in the frequency domain.

Knowing the parameters \(Z\) and \(Y\), the transfer matrix of the two-port corresponding to any length x can be derived by replacing (94) into (20),

\[
T_x = \begin{bmatrix}
\cosh(\Gamma x) & Z_c \sinh(\Gamma x) \\
Z_c^{-1} \sinh(\Gamma x) & \cosh(\Gamma x)
\end{bmatrix}. \tag{99}
\]

Equation (99) is the well known transfer matrix of a transmission line.

Whenever the following condition is met:

\[
|\Gamma x|^2 \ll 1. \tag{100}
\]

the first order development of the tanh function in (98) leads to the approximation

\[
\begin{cases}
Z_x \approx Z_c \Gamma x = Z_c x, \\
Y_x \approx Z_c^{-1} \Gamma x = Y_x. \tag{101}
\end{cases}
\]

The terms \(Z\) and \(Y\) now clearly appears to be respectively the shunt impedance and the open admittance per unit length. Hence, the transmission media is entirely described by either this pair \((Z, Y)\) of complex values per unit length, or the set made of the characteristic impedance \(Z_c\) and the propagation coefficient \(\Gamma\).

4.2. Evolution with length

It has been seen in the previous sections that the behavior of the system depends largely on the order of magnitude of \([Z_x, Y_x]\). It is therefore interesting to study the evolution of this norm with x. Replacing by the notation (95) in (96) leads to

\[
|Z_x Y_x| \approx \frac{\cosh(2ax) - \cos(4\pi x/\lambda)}{\cosh(2ax) + \cos(4\pi x/\lambda)}. \tag{102}
\]

For this analysis, it is assumed that \(\alpha \lambda \ll 1\). In addition, the domain for the parameter x is limited to \([0, x_{\text{max}}]\). Both conditions are not actually restrictive in most practical situations. Hence, the hyperbolic terms can be replaced by the first order development close to unit. In addition, the argument \(2ax\) varies slowly with x compared to the oscillations of the trigonometric terms whose argument is \(4\pi x/\lambda\). The numerator and denominator in (102) take simultaneously opposite extreme values at abscissa

\[
x_k = k \lambda/4 \quad (k \in \mathbb{N}). \tag{103}
\]

The even and odd indices k correspond to peaks of \(|Z_x Y_x|^{-1}\) and \(|Z_x Y_x|\), respectively. In the vicinity of these extrema, there is the development

\[
|Z_x Y_x|^{-1}(x_{2m} + \delta x) \quad \text{or} \quad |Z_x Y_x|(x_{2m+1} + \delta x) \approx (ax)^{-2} \left(1 + \left(\frac{2 \pi \delta x}{\alpha \lambda x}\right)^{-2}\right). \tag{104}
\]
Hence we obtain the following generalization of (101):

$$\left|Z_x Y_x\right| \ll 1 \Rightarrow \exists m \in \mathbf{Z}, \left\{\begin{array}{l} Z_x \approx Z_x - jZ_x m \pi, \\ Y_x \approx \overline{Y_x} - jZ_x^{-1} m \pi. \end{array}\right.$$

(111)

5. Conclusion

The power transfer coefficient through a two-port system has been expressed by means of two non-dimensional parameters: $\beta$ which is a characteristic parameter of the two-port, and $\sigma$ that depends on the external impedance load. A typical behavior of the relative power transfer coefficient has been put in evidence when $\beta$ is not too close to unit (see Figure 6).

A thorough analysis of what is interpreted as a half-wavelength line in most applications has been performed. It accounts also for the case of systems whose length is much smaller than the wavelength. Beyond the trivial result that such a two-port may have then little influence in the power transfer from one port to the other, we exhibited simple and accurate relations between input and output impedances, together with the precise conditions of validity for these approximations (see equation 86). Accordingly, the power transfer coefficient can be close to unit even though the loading impedance is different of the characteristic impedance, but the limits for which this property holds are asserted.

As far as passive, linear, symmetrical two-ports are involved, the results displayed in this paper can give interesting insights whatever the domain of application. Waveguides in acoustics and transmission line in the electrical domain are obvious fields of concern.

References


