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APPEARANCE OF A SOURCE/SINK LINE INTO A SWIRLING VORTEX

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Several mathematical models applied to tornadoes consist of exact and axisymmetric solutions of the steady and incompressible Navier–Stokes equations. These models studied by Serrin,[9] Goldshtik and Shtern[8] describe families of fluid motions vanishing at the ground and are restricted not to develop a source nor a sink near the vortex line. Therefore, Serrin showed that the flow patterns of the resulting velocity field may have some realistic characteristics to model the mature phase of the lifetime of a tornado, in comparison with atmospheric observations. On the other hand, no reason has been given to motivate the restriction of the absence of a source/sink vortex line. Therefore, we present here the construction and the analysis of a fluid motion driven by the vertical shear near the ground, the rate of the azimuthal rotation and by the intensity of a central source/sink line. We prove the local existence and uniqueness of a family of fluid motions, leading to the genesis of such source/sink lines inside a non-rotating updraft which does not develop, before perturbation, a source nor a sink.

Keywords: Exact solutions of Navier–Stokes; conical flows; vortex flows; source and sink line.

AMS Subject Classification: 76B47, 76U05

1. Introduction

The purpose of this paper is to suggest a new mathematical model applied to tornadoes, especially with the aim to describe some new features which were not included in the previous models. In particular, we want to include in this kind of fluid motion, the presence of a source/sink line into a swirling vortex, motivated — amongst other things — by the schema of the Minneapolis tornado (1986), proposed by Snow and Pauley[10] (see Fig. 1). There, we outline the existence of several regions in the flow, near the central axis, where the velocity normal to the axis may be nonzero.

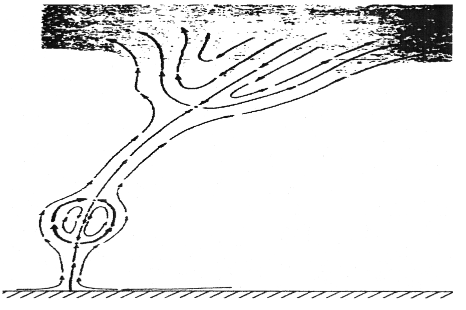


Fig. 1. Tornado of Minneapolis.[10]

So, realistic models applied to tornadoes, constituted by exact solutions of Navier–Stokes equations, must lead to this property of the velocity field. For this purpose, in this work, which also supplements the work we published[5] in 1998, we let the possibility of developing a parametrized source/sink line into a swirling vortex.

Because we choose conical flows to model tornadoes, it is clear that this way of modeling produces a singularity on the axis. The resulting behavior of the velocity suggests that other treatments must be done by dedicated tools — as asymptotic analysis — to detail the real situation inside the core of the tornado.

In addition, let us recall that neither the work of Serrin,[9] nor those of Goldshtik and Shtern[8] include the presence of a source/sink line inside the flows they modeled, but we did not find any reason which motivates this position, except the strength of the singularity existing in the model, which leads to other functional frames, as weight spaces, which are not needed in their models.

Before introducing the frame of our model, let us mention the main characteristics of conical flows applied to tornadoes. Most of investigations lead to stationary models which describe nice features of the mature phase of a tornado. Three authors clearly improved the knowledge in this field of research: Serrin[9] Goldshtik and Shtern.[8]

The approach of Serrin[9] was to model the main phase of a tornado by a family of exact solutions of the Navier–Stokes equations, which consists of a steady-state velocity field strictly respects the adherence condition at the ground and the requirement that the vortex line does not develop any sources nor sinks on the central axis. It means that no variation of the lateral mass flux is allowed at the neighborhood of the central axis. Consequently, the component of the velocity, corresponding to the radial outflow from the axis, is strictly zero at the vortex line. This last condition is not a realistic one as for a lot of tornadoes, such as the famous one in Minneapolis,[10] as we explained before.

In other words, streamlines must go out of the central line of the vortex if one wants to be able to generate the features of flows described above. Thus, the component of the velocity, corresponding to the radial outflow from the axis, must be nonzero at the vortex line.

This feature will be the main line of our model.

2. Mathematical Modeling

We introduce the spherical polar coordinates $(O; R, \alpha, \theta)$, where R denotes the radial distance from the origin, α the angle between the radius vector and the positive Z-axis, and θ is the meridian angle about the Z-axis. The positive Z-axis is then described by $\alpha = 0$, and the boundary plane $Z = 0$ by $\alpha = \pi/2$. The respective physical components of the velocity vector $\mathbf{V}$ in this coordinate system will be denoted u, v and w.

Our aim is to get a mathematical model which satisfies the following criteria:

(i) No-slip condition must be assumed when one goes to the ground ($Z = 0$).
(ii) Near the central axis ($\alpha = 0$), the behavior of the θ-component of the velocity field must approach the free vortex value.
(iii) The positive Z-axis must allow the presence of a source/sink line at the core of the flow.

To this end, we shall consider here steady-state fluid motions having the basic structure:

$$u(x, R) = \frac{F'(x)}{R}\,, \qquad v(x, R) = \frac{F(x)}{R\sqrt{1-x^2}}\,, \qquad w(x, R) = \frac{\Omega(x)}{R\sqrt{1-x^2}}\,, \tag{2.1}$$

where $r = R \sin \alpha$ is the distance to the Z-axis, $x = \cos \alpha$ and the prime denotes the derivative with respect to x.

This velocity field (2.1) has been considered by a lot of authors as Serrin, Goldshtik, Shtern and others (for a review, see Berker[2]). Standard studies showed that the structure (2.1) leads to streamlines which are connected with realistic streamlines as described by physical observations (see Ref. 5).

Indeed, one can remark that these streamlines present a characteristic form which corresponds to homothetic curves[2] for which the center is the origin O. In this way, this field can be proposed for modeling some atmospheric flows which govern the genesis of tornadoes.[4]

The first necessity is to derive appropriate conditions under which the basic motion (2.1) satisfies the steady and incompressible Navier–Stokes equations. Then, we find after some calculations that F and Ω are solution of the following nonlinear differential system of equations:

$$\begin{cases} \nu(1-x^2)F^{(4)} - 4\nu x F''' + FF''' + 3F'F'' = -\dfrac{2\Omega\Omega'}{1-x^2}\,, \\ \nu(1-x^2)\Omega'' + F\Omega' = 0\,, \end{cases} \qquad \forall\, x \in [0, 1[\,, \tag{2.2}$$

where ν denotes the kinematic viscosity of the fluid.

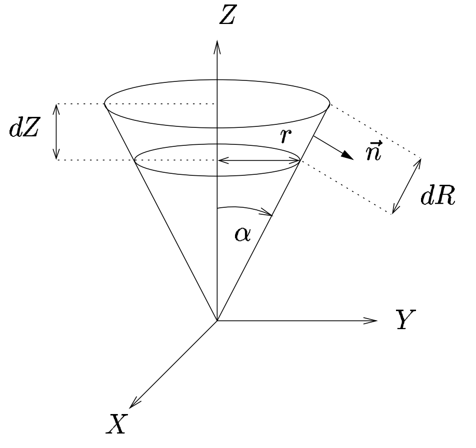


Fig. 2. Element of cone.

2.1. *Boundary conditions*

2.1.1. *Standard conditions*

According to conditions (i) and (ii), we must deal with the following boundary conditions applied to the functions F and Ω:

$$F(0) = F'(0) = \Omega(0) = 0\,, \tag{2.3}$$

$$\lim_{x \to 1^-} \Omega(x) = \Gamma\,, \tag{2.4}$$

where Γ is a given constant which drives the rotation of the flow around the Z-axis.

2.1.2. *Modeling of a source/sink line*

The presence of a source or a sink near the vertical central axis, i.e. condition (iii), is the reflection of a mass flux of the fluid which is nonzero when x goes to 1.

Then, we consider the mass flux $\dot{M}$ which goes out of a given element of cone (see Fig. 2), characterized by the angle α and the height dZ:

$$\dot{M} = \int_{\theta=0}^{2\pi} \int_{R}^{R+dR} \rho \mathbf{V} \cdot \mathbf{n}\, r d\theta dR\,, \tag{2.5}$$

where $\mathbf{n}$ is the normal unit vector of the element of the cone and ρ the density of the fluid. Due to the particular structure of the field of velocity (2.1), the flux $\dot{M}$ can be given by:

$$\dot{M} = \int_{\theta=0}^{2\pi} \int_{R}^{R+dR} \rho F(x) d\theta dR = \dot{m}\; dZ\,, \tag{2.6}$$

where $\dot{m}$ is the density of mass flux along the Z-axis and whose expression is:

$$\dot{m} = 2\pi\rho \frac{F(x)}{x}\,. \tag{2.7}$$

When we consider the limit such that the angle α of the cone tends to zero (x tends to 1), $\dot{m}$ is then the quantity of the fluid which is absorbing or ejecting by a unit element dZ of the vertical axis.

Therefore, our condition for the presence of a source/sink line along the Z-axis is:

$$\lim_{x \to 1^-} F(x) = \xi\,, \tag{2.8}$$

where ξ is a new parameter in the model which characterized the intensity of a sink when $\xi < 0$, and of a source when $\xi > 0$. These conditions guarantee that the line $\alpha = 0$ is a sink ($\dot{m} < 0$) or a source ($\dot{m} > 0$), for a given intensity ξ.

Remark. We notice that problem (2.2) is a system of ordinary differential equations of the sixth order. In addition, in this section, we formulated five boundary conditions motivated by physical properties of the flow we wanted to model. Then an extra degree appears. With the exception of the parameters of the boundary conditions Γ and ξ, we must deal with a one-parameter family of solutions of Navier–Stokes equations. This extra degree allowed one to propose a mathematical model for tornado genesis by a bifurcation process,[4] where it played the role of the bifurcation parameter.

2.2. *Transformation of the mathematical problem*

We are now in a position to give a complete formulation of the mathematical problem. Taking into account the different boundary conditions we presented in the previous section, we first give several transformations of the integration of the fourth differential equation of the system (2.2). Three integrations by parts, on the one hand, and using the boundary conditions of the previous section, on the other hand, lead us to rewrite this equation on F as follows:

$$2\nu(1 - x^2)F' + 4\nu x F + F^2 = H(x)\,, \tag{2.9}$$

with

$$H(x) = -2\int_0^x \frac{(x-t)(1-xt)}{(1-t^2)^2}\Omega^2(t)dt + Px^2 + 2Bx\,, \tag{2.10}$$

where P and B are two constants of integration. Now, we are interested in the asymptotic behavior of this equation in a neighborhood of $x = 1$. As F takes the finite nonzero value ξ when x goes to 1, we find that

$$(1 - x^2)F' \to 0 \quad \text{as} \quad x \to 1\,. \tag{2.11}$$

Taking $x = 1$ in Eq. (2.9) gives then the relation:

$$4\nu\xi + \xi^2 = -2\int_0^1 \frac{\Omega^2(t)}{(1+t)^2}dt + P + 2B\,, \tag{2.12}$$

and the second member H can be written in the following expression:

$$H(x) = -P(x - x^2) + (4\nu\xi + \xi^2)x + 2(1-x)^2 \int_0^x \frac{t\Omega^2(t)}{(1-t^2)^2} dt + 2x \int_x^1 \frac{\Omega^2(t)}{(1+t)^2} dt\,, \tag{2.13}$$

The new mathematical formulation of the problem is then:

$$(\mathcal{P}_\xi)_1 \begin{cases} 2\nu(1-x^2)F' + 4\nu x F + F^2 = H(x)\,, \\ \nu(1-x^2)\Omega'' + F\Omega' = 0\,, \end{cases} \quad \forall\, x \in [0,1[\,, \tag{2.14}$$

where the second member H is given by (2.13), the functions F and Ω are submitted to the boundary conditions:

$$F = \Omega = 0 \quad \text{at} \quad x = 0\,, \quad \Omega \to \Gamma \quad \text{as} \quad x \to 1\,. \tag{2.15}$$

A last change of variables defined by:

$$F = 2\nu(1-x^2)f\,, \qquad k = \frac{1}{2\nu}\,, \tag{2.16}$$

gives:

$$(P_\xi)_2 \begin{cases} f' + f^2 = k^2 \dfrac{H(x)}{(1-x^2)^2}\,, \\ \Omega'' + 2f\Omega' = 0\,, \end{cases} \quad \forall\, x \in [0,1[\,, \tag{2.17}$$

with the boundary conditions:

$$f = \Omega = 0 \quad \text{at} \quad x = 0\,, \quad \Omega \to \Gamma \quad \text{as} \quad x \to 1\,. \tag{2.18}$$

Solutions (f, Ω) of the above problem are a family of solutions depending on the following three parameters: P, Γ and ξ. The parameter P is the extra degree we mentioned in the remark of the last section. Its physical interpretation will be detailed in Sec. 3. There, we will see that — amongst other things — it characterizes the pressure at the ground in comparison with the pressure of a free vortex. Another way to exhibit physical meaning of parameter P — the intensity of the vertical shear near the ground — was given in Ref. 3. On the other hand, parameters Γ and ξ are the driving parameters of the flow.

2.3. *Existence and uniqueness of solutions*

This section is devoted to establish the existence and the uniqueness of a couple of solutions (f, Ω) of problem $(\mathcal{P}_\xi)_2$ by a method of perturbation of the parameters (Γ, λ) near zero, where λ is defined by:

$$\lambda = \left(\frac{2\xi}{k} + \xi^2\right)\,. \tag{2.19}$$

Precisely, when $\lambda = 0$ (and $\xi = 0$) on the one hand, and $\Gamma = 0$, on the other hand, we know[3] that there exists a solution of problem $(\mathcal{P}_\xi)_2$ which depends on a given parameter P_0 when the condition

$$k^2 P_0 < \chi^2, \qquad (\chi \approx 2.85)\,, \tag{2.20}$$

is satisfied.

In this case, let us introduce $(f_{P_0}, \Omega_0 \equiv 0)$ this particular solution of the problem $(\mathcal{P}_\xi)_2$, to focus on its dependence with the choice of the parameter P_0. This kind of solutions has a smooth behavior at the neighborhood of $x = 1$, in comparison with the solutions of problem $(\mathcal{P}_\xi)_2$ which take into account nonzero values of ξ, characterizing the presence of the source/sink line.

More precisely, we showed[3] that, for $\Omega \equiv 0$ and $\xi = 0$, the solution f behaves as $\log(1-x)$ as x tends to 1, whereas the solution f corresponding to $\Omega \neq 0$ and $\xi \neq 0$ will be more singular on this same neighborhood. We will see in the next section that the singularity therefore has the form of $1/(1-x)$. We must take into account the strength of this singularity for further functional frame. Let us now introduce the function y defined by:

$$y(x) = \exp\left(\int_0^x f(t)dt\right)\,, \tag{2.21}$$

where f is a solution of the problem $(\mathcal{P}_\xi)_2$.

Then, one can show that (y, Ω) is a solution of:

$$y(x) = 1 + k^2 \int_0^x \frac{(x-t)H(t)y(t)}{(1-t^2)^2}dt\,, \tag{2.22}$$

$$\Omega(x) = \Gamma\left(\int_0^x y^{-2}(t)dt\right)\left(\int_0^1 y^{-2}(t)dt\right)^{-1}\,, \tag{2.23}$$

where function H is given by (2.13). Finally, we define the functional ϕ by:

$$\phi : C^0_\alpha([0,1]) \times C^0([0,1]) \times \mathbb{R}^2 \longrightarrow C^0_\alpha([0,1]) \times C^0([0,1])\,,$$

$$(y, \Omega; \Gamma, \lambda) \longmapsto (\tilde{y}, \tilde{\Omega})\,,$$

$$\tilde{y}(x) = y(x) - 1 - k^2 \int_0^x \frac{(x-t)H(t)y(t)}{(1-t^2)^2}dt\,, \tag{2.24}$$

$$\tilde{\Omega}(x) = \Omega(x) - \Gamma\left(\int_0^x y^{-2}(t)dt\right)\left(\int_0^1 y^{-2}(t)dt\right)^{-1}\,, \tag{2.25}$$

where the weight space $C^0_\alpha([0,1])$ is defined by:

$$C^0_\alpha([0,1]) \equiv \{y : [0,1[\longrightarrow \mathbb{R}; (1-x)^\alpha y \in C^0([0,1])\}\,, \tag{2.26}$$

for all $\alpha > 0$.

The space $C^0_\alpha([0,1])$ is a Banach space with respect to the norm:

$$\|y\|_\alpha \equiv \sup_{0\le x\le 1} (1-x)^\alpha |y(x)| \,. \tag{2.27}$$

The following lemmas guarantee, on the one hand, that the functional frame correctly defines the operator ϕ, and on the other hand, that the differential of ϕ presents nice characteristics at the point $(y_0, \Omega_0 \equiv 0; \Gamma_0 \equiv 0, \lambda_0 \equiv 0)$ with $\xi_0 = 0$, satisfying

$$\phi(y_0, \Omega_0 \equiv 0; \Gamma_0 \equiv 0, \lambda_0 \equiv 0) = 0\,,$$

where y_0 corresponds to the solution f_{P_0} of the problem $(\mathcal{P}_\xi)_2$. Moreover, it will be shown that the differential $D_X\phi(y_0, \Omega_0; \Gamma_0, \lambda_0)$ with respect to $X = (y, \Omega)$, is a Fredholm operator.

Lemma 1. *Let* $(y, \Omega) \in C^0_\alpha([0,1]) \times C^0([0,1])$, *then* $(\tilde{y}, \tilde{\Omega}) \equiv \phi(y, \Omega; \Gamma, \lambda)$ *belongs to* $C^0_\alpha([0,1]) \times C^0([0,1]), \forall\, \alpha > 0$.

Lemma 2. *The functional* ϕ *is differentiable and its differential with respect to* $X = (f, \Omega)$ *is given by* $D_X\phi(y_0, \Omega_0; \Gamma_0, \lambda_0) \cdot (u, w) \equiv (\tilde{u}, \tilde{w})$, *where*

$$\tilde{u}(x) = u(x) + k^2 P_0 \int_0^x \frac{(x-t)tu(t)}{(1-t)(1+t)^2} dt\,, \quad \tilde{w}(x) = w(x)\,. \tag{2.28}$$

Lemma 3. *Let* $0 < \alpha < 3/2$. *The kernel of the linear operator* $D_X\phi(y_0, \Omega_0; \Gamma_0, \lambda_0)$ *reduces to the zero space.*

Proof. The equation of the kernel is a Volterra equation of the second kind, with a kernel which belongs to $L^2(0,1) \times L^2(0,1)$, for all $\alpha \in]0, 3/2[$. So, standard theory (see Tricomi[11]), applied to this type of integral equations, leads to the unique solution $u = 0$. □

Theorem 1. *Let* P_0 *satisfying* (2.20) *and* $(y_0, \Omega_0 \equiv 0)$, *the corresponding solutions of problem* (2.23)–(2.23), *with* $\xi_0 = 0$ *and* $\Gamma_0 = 0$. *Therefore, there exists a neighborhood of* $(y_0, \Omega_0 \equiv 0; \Gamma_0 \equiv 0, \lambda_0 \equiv 0)$, $(\xi_0 = 0)$, *such that the equation* $\phi(y, \Omega; \Gamma, \lambda) = 0$ *admits one and only one continuous branch of solutions* (y, Ω), *which belongs to* $C^0_\alpha([0,1]) \times C^0([0,1])$, *for all* α *satisfying* $0 < \alpha < 1$.

Proof. This directly results from the application of the implicit function theorem. To this end, let us observe that the operator $D_X\phi(y_0, \Omega_0; \Gamma_0, \lambda_0)$ is a perturbation of the identity. Moreover, we can show that this perturbation is compact on $C^0_\alpha([0,1]) \times C^0([0,1])$ (see Ref. 5). Therefore, $D_X\phi(y_0, \Omega_0; \Gamma_0, \lambda_0)$ is a Fredholm operator and its index is zero.[1] So, its codimension is equal to zero because we saw in Lemma 3 that its kernel is also reduced to zero. Then, we can conclude that $D_X\phi(y_0, \Omega_0; \Gamma_0, \lambda_0)$ is a bijection on $C^0_\alpha([0,1]) \times C^0([0,1])$. This property allows us to apply the implicit function theorem. □

2.4. *Structure of solutions*

This section is devoted to the analysis of the structure and the behaviors of the solutions of the problem $(\mathcal{P}_\xi)_2$. To this end, we proceed to a normalization of the variables and the parameters as follows: $F = \Gamma\tilde{F}$, $\Omega = \Gamma\tilde{\Omega}$, $P = \Gamma^2\tilde{P}$ and $\xi = \Gamma\tilde{\xi}$. Then, we get the normalized formulation on the quantities $\tilde{}$:

$$(\tilde{\mathcal{P}}_\xi)_1 \quad \begin{cases} \dfrac{1}{\mathcal{R}e}\left(2(1-x^2)\tilde{F}' + 4x\tilde{F}\right) + \tilde{F}^2 = \tilde{H}(x)\,, \\ \dfrac{1}{\mathcal{R}e}(1-x^2)\tilde{\Omega}'' + \tilde{F}\tilde{\Omega}' = 0\,, \end{cases} \qquad \forall\, x \in [0,1[\,, \tag{2.29}$$

in which $\mathcal{R}e = \frac{\Gamma}{\nu}$ and $\tilde{H}$ corresponds to the function H written with the normalized quantities, i.e.

$$\tilde{H}(x) = -\tilde{P}(x-x^2) + \left(\frac{4}{\mathcal{R}e}\tilde{\xi} + \tilde{\xi}^2\right)x + 2(1-x)^2\int_0^x \frac{t\tilde{\Omega}^2(t)}{(1-t^2)^2}dt + 2x\int_x^1 \frac{\tilde{\Omega}^2(t)}{(1+t)^2}dt \tag{2.30}$$

and the functions $\tilde{F}$ and $\tilde{\Omega}$ are submitted to the boundary conditions

$$\tilde{F} = \tilde{\Omega} = 0 \quad \text{at} \quad x = 0\,, \quad \tilde{\Omega} \to 1 \quad \text{as} \quad x \to 1\,. \tag{2.31}$$

A last change of variables defined by:

$$\tilde{F} = \frac{2}{\mathcal{R}e}(1-x^2)\tilde{f}\,, \tag{2.32}$$

gives

$$(\tilde{\mathcal{P}}_\xi)_2 \begin{cases} \tilde{f}' + \tilde{f}^2 = \dfrac{\mathcal{R}e^2\tilde{H}(x)}{4(1-x^2)^2}\,, \\ \tilde{\Omega}'' + 2\tilde{f}\tilde{\Omega}' = 0\,, \end{cases} \qquad \forall\, x \in [0,1[\,, \tag{2.33}$$

with the boundary conditions:

$$\tilde{f} = \tilde{\Omega} = 0 \quad \text{at} \quad x = 0\,, \quad \tilde{\Omega} \to 1 \quad \text{as} \quad x \to 1\,. \tag{2.34}$$

The problem $(\tilde{\mathcal{P}}_\xi)_2$ describes a three-parameter ($\tilde{P}$, $\mathcal{R}e$, $\tilde{\xi}$) family of solutions. In the following, and to keep a mathematical formulation as light as possible, the normalized quantities will be written without the notation $\tilde{}$.

Now, rather than considering the parameter ξ, we introduce the quantity P_ξ defined by:

$$P_\xi = \frac{4}{\mathcal{R}e}\xi + \xi^2\,. \tag{2.35}$$

We showed the existence and uniqueness of the solution in a neighborhood of $\xi = 0$ (i.e. $P_\xi = 0$), but the existence of solutions is not possible for all values of the parameter P_ξ. More precisely, we have:

Theorem 2. *For any values of ξ strictly smaller than $\xi^\star = -2/\mathcal{R}e$, the problem $(\tilde{\mathcal{P}}_\xi)_2$ has no solution.*

Proof. We integrate successively, twice by parts the second equation of the problem $(\tilde{\mathcal{P}}_\xi)_2$ to obtain the expression of Ω:

$$\Omega(x) = C_0 \int_0^x \exp\left(-2\int_0^t f(s)ds\right) dt + C_1\,, \tag{2.36}$$

where C_0 and C_1 are two constants of integration which are determined by the boundary conditions (2.34). Therefore,

$$\Omega(x) = \frac{1}{\gamma}\int_0^x \exp\left(-2\int_0^t f(s)ds\right) dt\,, \tag{2.37}$$

where γ is strictly positive, and is given by

$$\gamma = \int_0^1 \exp\left(-2\int_0^t f(s)ds\right) dt\,. \tag{2.38}$$

We will prove that the evaluation of γ is not possible for all the values of the parameter ξ. We examine to this end the behavior of the function f near $x = 1$.

The formula (2.32) enables us to interpret the condition (2.8) of a source/sink line, as the mathematical behavior of the function f near $x = 1$, as follows:

$$f \underset{x\to 1^-}{\sim} \frac{\mathcal{R}e\,\xi}{4(1-x)}\,, \tag{2.39}$$

Therefore, for any real number ε strictly positive, there exists x_0 inside the interval $]0,1[$ such that, for all x greater than x_0, we have:

$$\frac{\eta^-}{2(1-x)} < f(x) < \frac{\eta^+}{2(1-x)}\,, \tag{2.40}$$

where

$$\left.\begin{aligned} \eta^- &= \frac{\mathcal{R}e\,\xi}{2}(1+\varepsilon) \\ \eta^+ &= \frac{\mathcal{R}e\,\xi}{2}(1-\varepsilon) \end{aligned}\right\} \quad \text{for} \quad \xi < 0 \quad \text{and} \quad \left.\begin{aligned} \eta^- &= \frac{\mathcal{R}e\,\xi}{2}(1-\varepsilon) \\ \eta^+ &= \frac{\mathcal{R}e\,\xi}{2}(1+\varepsilon) \end{aligned}\right\} \quad \text{for} \quad \xi > 0\,.$$

Integrating the double inequality (2.40) on $[x_0, x]$, one can obtain:

$$-2I_0 + \eta^+ \ln\left(\frac{1-x}{1-x_0}\right) < -2\int_0^x f(t)dt < -2I_0 + \eta^- \ln\left(\frac{1-x}{1-x_0}\right)\,, \tag{2.41}$$

where $I_0 = \int_0^{x_0} f(t)dt$.

Finally, taking the exponential of each term, we get:

$$\exp(-2I_0)\left(\frac{1-x}{1-x_0}\right)^{\eta^+} < \exp\left(-2\int_0^x f(t)dt\right) < \exp(-2I_0)\left(\frac{1-x}{1-x_0}\right)^{\eta^-}\,. \tag{2.42}$$

The double inequality (2.42) will be taken into account to control the integral (2.38) which defines γ. To this end, we cut the concerning integral between $[0, x_0]$ (named J_0) and $[x_0, x]$. By using (2.42) and taking the limit when x goes to 1, we find:

$$\gamma > J_0 + \frac{(1-x_0)\exp(-2I_0)}{1+\eta^+} \lim_{x\to 1^-} \left[1 - \left(\frac{1-x}{1-x_0}\right)^{1+\eta^+}\right], \tag{2.43}$$

$$\gamma < J_0 + \frac{(1-x_0)\exp(-2I_0)}{1+\eta^-} \lim_{x\to 1^-} \left[1 - \left(\frac{1-x}{1-x_0}\right)^{1+\eta^-}\right], \tag{2.44}$$

Therefore, if the right-hand side of (2.43) diverges, which means that $1+\eta^+$ is strictly negative, then the condition $\Omega(1) = 1$ cannot be satisfied. So, there appears a critical value $\xi^\star$ which corresponds to $1+\eta^+ = 0$, such that the problem $(\tilde{\mathcal{P}}_\xi)_2$ has no solution. This value depending on each ε, the critical value $\xi^\star$ is then given by taking the limit when ε goes to 0:

$$\xi^\star = \frac{-2}{\mathcal{R}e}\,. \tag{2.45}$$

Then, for all $\xi \geq \xi^\star$, the limit defining γ exists. It is the reason why, in the sequel, we will consider only the values of the parameter ξ such that the condition $\xi \geq \xi^\star$ is satisfied. □

The structure of the solutions of the integro-differential system $(\tilde{\mathcal{P}}_\xi)_2$ depends essentially on the function H defined by (2.30). Then, we present now the main results in the two following lemmas, concerning the behavior of this function.

Lemma 4. *Let us introduce I_Ω defined by: $I_\Omega = \int_0^1 \frac{\Omega^2(t)}{(1+t)^2}dt$. This integral belongs to the interval $]0, 1/2[$ and the function H satisfies the boundary conditions:*

$$\begin{aligned} H(0) &= 0\,,\\ H'(0) &= 2I_\Omega - P + P_\xi\,,\\ H''(0) &= 2P\,, \end{aligned} \quad \text{and} \quad \begin{aligned} H(1) &= P_\xi\,,\\ H'(1) &= P - 1 + P_\xi\,. \end{aligned}$$

Moreover, the third derivative H''' is strictly negative on the entire interval $]0, 1[$.

Proof. Each step of this lemma is done as in Serrin's lemma.[9] Because the derivative of function Ω in our formulation still stays positive, the function H''' is also negative (see the expression of H''' in Serrin's model which is exactly the same as in our model). □

Lemma 5. *The function H has five possible behaviors according to the values of the parameters P, $\mathcal{R}e$ and P_ξ.*

(1) *If $P_\xi \in\,]\sup(1-P, P_\xi^\star), 0[$, or $P_\xi \in]P_\xi^\star, P - 2I_\Omega[$ with $P \leq 0$, then H is negative everywhere.*

(2) *If* $P_\xi \in]0, P - 2I_\Omega[$, *then* H *is first negative, has a single zero and is positive thereafter.*
(3) *If* $P_\xi \in [\sup(0, P - 2I_\Omega), +\infty[$, *then* H *is positive everywhere.*
(4) *If* $P_\xi \in]\sup(P_\xi^\star, P - 2I_\Omega), 0[$, *then* H *is first positive, has a single zero and then is negative.*
(5) *If* $P_\xi \in]P_\xi^\star, \inf(0, 1 - P, P - 2I_\Omega)[$ *with* $P > 0$, *then, either* H *is negative everywhere, or* H *is first negative and changes sign twice.*

Proof. First of all, Lemma 4 gave us the sign of H''', which is strictly negative. This implies that the function H'' decreases strictly monotonically on the entire interval $]0, 1[$. Therefore, H'' has at most a single zero on $]0, 1[$. Thus, H is either convex, or concave, or first convex then concave.

We will only present the first case with every details, which is when H is convex on the entire interval $]0, 1[$ (the other cases might be consulted in Ref. 7). In this case, H'' is strictly positive on $]0, 1[$, and H' increases strictly monotonically. The derivative of H has then at most a single zero on $]0, 1[$.

Depending on the sign of the limits of H' at $x = 0$ and $x = 1$, H' is either positive, or first negative and after positive, or negative everywhere.

We are now in position to conclude the behavior of H in these three cases:

- If $H'(0)$ is positive, then H' is strictly positive on $]0, 1[$. Now, because H is zero at $x = 0$, we conclude that H is strictly positive on $]0, 1]$.
- If $H'(0)$ is negative and $H'(1)$ is positive, then H' is first negative, has a zero at x_0 $(0 < x_0 < 1)$, then is positive. The function H decreases then first, has a minimum at x_0 and increases thereafter. As H is zero at $x = 0$, H is negative at a neighborhood of $x = 0$. If $H(1)$ is negative, then H keep a constant sign which is negative, but if $H(1)$ is positive, then H is first negative, has a zero and stays positive thereafter.
- If $H'(0)$ and $H'(1)$ are both negative, then H' is negative everywhere. In this case, H decreases strictly from its zero value at $x = 0$, and stays strictly negative on $]0, 1]$. □

Figure 3 gives the summary of the behaviors of the function H according to the previous lemma. As the parameter P_ξ defined by (2.35) has the minimum value $P_\xi^\star = -4/\mathcal{R}e^2$, we represent on the parametric half-plane $(P_\xi \geq P_\xi^\star, P)$, the zones that characterize the five possible behaviors of function H.

- Zone 1 defined by $P_\xi \in]\sup(1 - P, P_\xi^\star), 0[$, or $P_\xi \in]P_\xi^\star, P - 2I_\Omega[$ with $P \leq 0$, corresponds to H negative.
- Zone 2 defined by $P_\xi \in]0, P - 2I_\Omega[$, corresponds to H first negative then positive.
- Zone 3 defined by $P_\xi \in [\sup(0, P - 2I_\Omega), +\infty[$, corresponds to H positive.
- Zone 4 defined by $P_\xi \in]\sup(P_\xi^\star, P - 2I_\Omega), 0[$, corresponds to H first positive then negative.

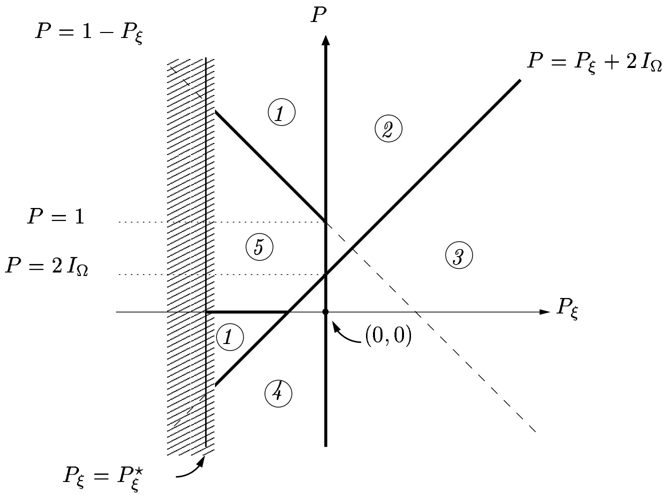


Fig. 3. Behaviors of function H in the plane of parameters (P_ξ, P).

- Zone 5 defined by $P_\xi \in]P_\xi^\star, \inf(0, 1-P, P-2I_\Omega)[$ with $P > 0$, corresponds to two possible behaviors of H: either H is negative, or H is first negative then changes sign twice.

In this representation, the boundary $P_\xi = 0$ plays a particular role because, for this particular value of P_ξ, the function H becomes equal to the function treated by Serrin without the presence of a source nor a sink.

The five possible behaviors of the function H lead to several behaviors of the component v defined by the function F, solution of the first differential equation of (2.29). Therefore, we give now the results qualifying the function f, solution of (2.33):

Theorem 3. *Let f be a solution of the problem $(\tilde{\mathcal{P}}_\xi)_2$, then*

(1) *if (P_ξ, P) belongs to zone* 1, *then the function f is negative.*
(2) *if (P_ξ, P) belongs to zone* 2 *and $\xi \in]0, +\infty[$, then the function f is first negative, has a single zero and is positive thereafter.*
(3) *if (P_ξ, P) belongs to zone* 3 *and $\xi \in]0, +\infty[$, then the function f is positive.*
(4) *if (P_ξ, P) belongs to zone* 4, *then the function f is first positive, has a single zero and is negative thereafter.*
(5) *if (P_ξ, P) belongs to zone* 5, *then the function f is either everywhere negative, or first negative then changes sign twice.*

Proof. For a first step we show how the sign of H determines whose of f. By multiplying the Riccati equation of the system (2.33) by the integrant factor

$\exp(\int_0^x f(t)dt)$, we obtain

$$f(x) = \frac{\mathcal{R}e^2}{4} \int_0^x \frac{H(t)}{(1-t^2)^2} \exp\left(-\int_t^x f(s)ds\right) dt\,. \tag{2.46}$$

But, we proved in Lemma 5 that the function H has five possible behaviors which can be summarized by: Either H has a constant sign, or H changes only one time its sign, or H is first negative, then changes sign twice. We conclude by (2.46) that for each case, the sign of the function f can be deduced from:

- If H has a constant sign, then f also has a constant sign (the same as H).
- If H changes sign one time, then either f has a constant sign (those of H in the neighborhood of $x = 0$), or f has the same behavior as H.
- If H is negative and changes sign twice, then either f is negative, or f is first negative and positive thereafter, or f is exactly as H.

Moreover, the sign of f in the neighborhood of $x = 1$ is given by the sign of ξ, because of the equivalence (2.39) of f. Therefore, we are interested in each zone which is characteristic of the behavior of H:

If P and ξ are such that the point (P_ξ, P) is in zone 1, then H is negative and f too.

If P and ξ are such that the point (P_ξ, P) is in zone 2, then H is negative and positive thereafter. In this zone, the parameter P_ξ is positive, then *a priori*, either ξ belongs to $]-\infty, -4/\mathcal{R}e[$, or ξ belongs to $]0, +\infty[$. But, it is necessary to consider $\xi \geq \xi^\star$ (see Theorem 2), then only the case ξ positive is available. As the function f is positive as x tends to 1, and has the same sign as H in the neighborhood of $x = 0$, the function f is then first negative and positive thereafter.

If P and ξ are such that the point (P_ξ, P) is in zone 3, then H is positive and f too. But as f is positive, it is necessary to have ξ positive.

If P and ξ are such that the point (P_ξ, P) is in zone 4, then H is positive and negative thereafter. In this zone, the parameter P_ξ is negative, so *a priori* $\xi \in]-4/\mathcal{R}e, 0[$. But ξ must be greater than $\xi^\star$, so it means that this zone is available only for $\xi \in [\xi^\star, 0[$, and the function f can only be first positive, then negative.

If P and ξ are such that the point (P_ξ, P) is in zone 5, then H is either negative, or negative and changes sign twice. So in this zone, f is either everywhere negative, or negative then positive, or negative and then changes sign twice. But, because the parameter P_ξ is negative, ξ belongs to $]\xi^\star, 0[$ and f must be negative in the neighborhood of $x = 1$. So, either f is everywhere negative, or f is first negative then changes sign twice. □

Later in the paper, we will present the visualization of each type of flows resulting of this analysis.

2.5. *Behavior of solutions*

We focus in this section, the analysis of the limit behavior of functions F and Ω at $x = 1$, near the Z-axis. We will use the following results to determine how the field of velocity and the pressure behave in the core of the flow.

The boundary conditions (2.8) modeling the presence of a sink or of a source near the central axis, and those which models that the flow behaves as a free vortex (2.4), define the whole behavior of the functions F and Ω as x tends to 1, and therefore, the components v and w of the velocity field.

But the behavior of the derivatives F' and Ω' at the same neighborhood ($x = 1$) is not so immediate. To this end, we show the following theorem:

Theorem 4. *If ξ is positive, then Ω' has a limit value which is zero as x tends to* 1, *but when ξ belongs to the interval $[\xi^\star, 0[$, Ω' is not bounded.*

Proof. To guarantee that the function Ω satisfies the boundary condition $\Omega(1) = 1$, we have to consider the values of ξ in the interval $[\xi^\star, +\infty[$. In this case, we consider the inequalities (2.43)–(2.44), which are available for all ε strictly positive. If we choose ε in $]0, 1[$, the coefficients η^- and η^+ have the same sign as ξ. Then, we can conlude that in the case $\xi > 0$ ($\eta^- > 0$), the function Ω' is positive or null on the entire interval $[0, 1]$, and is lower than a function which goes to 0 as x goes to 1; that proves that $\Omega'(1) = 0$.

In the case $\xi < 0$ ($\eta^+ < 0$), the function Ω' is greater than a positive and not bounded function, as x goes to 1. The function Ω' is also not bounded in the neighborhood of $x = 1$, for negative values of ξ, but greater than the critical value $\xi^\star$. □

We did not succeed to determine completely the behavior of the radial component u of the velocity field, defined by the function F'. We essentially show the following result:

Theorem 5. *If ξ is positive, the derivative F' has a finite limit value as x tends to* 1 *and is equal to*:

$$F'(1) = \frac{1}{2\xi}[P - 1 + \xi^2]\,. \tag{2.47}$$

Proof. First of all, we give the expression of F' from Eq. (2.29) of the problem $(\tilde{\mathcal{P}}_\xi)_1$:

$$F' = \frac{\mathcal{R}e}{2(1+x)}\left[\frac{H - (F^2 + \frac{4}{\mathcal{R}e}xF)}{1-x}\right]. \tag{2.48}$$

Because the quantity into the brackets is undetermined when x goes to 1, (the form is $0/0$), we use the Hospital's rule and we obtain:

$$\lim_{x\to 1^-} F' = -\frac{\mathcal{R}e}{4}\lim_{x\to 1^-}\left[H' - \frac{4}{\mathcal{R}e}F - 2F'\left(\frac{2x}{\mathcal{R}e} + F\right)\right]. \tag{2.49}$$

Now, at this stage of the proof, we take into account the sink/source condition and the value of H' at $x = 1$, given by Lemma 4. So, we can write:

$$\lim_{x\to 1^-} F' = -\frac{\mathcal{R}e}{4}(P - 1 + \xi^2) + \left(1 + \frac{\mathcal{R}e\,\xi}{2}\right)\lim_{x\to 1^-} F' \,. \tag{2.50}$$

If the function F' is bounded, relation (2.50) gives us the limit value of F' as x tends to 1. But, we still do not know if F' stays bounded, so we will investigate in order to determine conditions which can guarantee this situation. If one derives Eq. (2.29) on F, and takes the limit as x tends to 1, by considering the boundary conditions on F and H', we get:

$$\frac{1}{\mathcal{R}e}\lim_{x\to 1^-}(1-x^2)F'' + \xi \lim_{x\to 1^-} F' = \frac{1}{2}(P - 1 + P_\xi) - \frac{2}{\mathcal{R}e}\,\xi = \frac{1}{2}(P - 1 + \xi^2)\,, \tag{2.51}$$

in which we substitute the limit value (2.50) of F' as x tends to 1:

$$\frac{2}{\mathcal{R}e}\lim_{x\to 1^-}(1-x)F'' + \xi\left(1 + \frac{\mathcal{R}e\,\xi}{2}\right)\lim_{x\to 1^-} F' = \frac{1}{2}(P - 1 + \xi^2)\left(1 + \frac{\mathcal{R}e\,\xi}{2}\right). \tag{2.52}$$

Now suppose that F' diverges when x goes to 1, then the function $(1-x)F''$ must also not be bounded at the neighborhood of $x = 1$, because (2.52) must be finite on both sides. So, the only way to guarantee this situation is given by:

$$\xi\left(1 + \frac{\mathcal{R}e\,\xi}{2}\right) < 0\,. \tag{2.53}$$

When we introduce the critical value $\xi^\star$ in this relation, we show that:

$$\xi(\xi - \xi^\star) > 0 \Rightarrow F' \text{ bounded when } x \to 1\,, \tag{2.54}$$

that completes the proof of the theorem since we consider values of ξ greater than $\xi^\star$. □

3. Physical Interpretation

3.1. *Behavior of the physical variables*

The study of the functions F and Ω we detailed in the previous section, leads us to give physical interpretations about the flow near the axis. In the sequel, parameters P, P_ξ (or ξ) and the integral I_Ω, will denote no dimensioned parameters but physical variables will still be dimensioned. A simple way to interpret the results of the previous sections, as the behaviors of the physical variables, is to consider the cylindrical components (V_r, V_θ, V_Z) of the field of velocity $\mathbf{V}$. Then, we are in a position to give the behavior of the velocity near the axis, when x goes to 1. The boundary conditions (2.4) and (2.8) we considered, lead to a singularity of the radial and azimuthal components of the velocity:

$$\begin{aligned} V_r(r,Z) &\underset{r\to 0}{\sim} \frac{\Gamma\xi}{r}\,, \\ V_\theta(r,Z) &\underset{r\to 0}{\sim} \frac{\Gamma}{r}\,. \end{aligned} \tag{3.55}$$

On the other hand, the vertical component of the velocity is entirely determined and bounded on the axis, in the case of a source line ($\xi > 0$), and is equal to

$$V_Z(r=0,Z) = \frac{\Gamma}{2\xi Z}[P - 1 - \xi^2]\,. \tag{3.56}$$

Furthermore, in the case of a sink line ($\xi < 0$), we proved that ξ must belong to $[\xi^\star, 0[$, and we did not exhibit the behavior of the vertical velocity near the central axis. However, we observed numerically that it could be either bounded or not. Several authors as Goldshtik and Shtern[8] treated in the past the case of a bounded vertical component. We do allow a singularity in this component, as in the Serrin model[9] and we will motivate this position after the following analysis.

Let us recall that the reduced pressure is given by:

$$\bar{p}(x,R) = \frac{\Gamma^2 \pi(x)}{R^2(1-x^2)}\,, \tag{3.57}$$

where the expression of the nondimensioned function π is:

$$\pi(x) = -\frac{1}{2}\left\{F^2(x) + (1-x^2)\left[P + 2\int_0^x \frac{t\Omega^2(t)}{(1-t^2)^2}dt\right] - \frac{2}{\mathcal{R}e}(1-x^2)F'(x)\right\}. \tag{3.58}$$

With this expression, we get simply the behavior of the pressure near the boundaries:

$$\bar{p}(r, Z=0) = -\frac{\Gamma^2 P}{2r^2} \quad \text{and} \quad \bar{p}(r,Z) \underset{r\to 0}{\sim} -\frac{\Gamma^2}{2r^2}(1+\xi^2) \tag{3.59}$$

So, we can deal with a first physical interpretation of the parameters P and ξ. The pressure at the ground is exactly P times the pressure developed by a free vortex (see Ref. 9). Moreover, the presence of the source/sink line contributes to increase the depression induced by the rotation at the neighborhood of the vortex axis.

On the other hand, the main physical interpretation of the parameters is given by the expressions of the stresses, especially by the horizontal shear inside the flow on the central axis $\mathcal{T}$, and the vertical shear near the ground τ_r, which expressions are:

$$\mathcal{T} = \frac{2\pi\mu}{Z}\lim_{x\to 1^-}[(1-x^2)F''(x) + 2F(x)] \quad \text{and} \quad \tau_r = \frac{\mu F''(0)}{r^2}\,. \tag{3.60}$$

In the Serrin's model ($\xi = 0$), one can prove that these stresses are given by:

$$\mathcal{T} = \frac{\rho\pi\Gamma^2}{Z}(P-1) \quad \text{and} \quad \tau_r = \frac{\rho\Gamma^2}{2r^2}(2I_\Omega - P)\,. \tag{3.61}$$

When the shear τ_r near the ground is positive, the fluid is ejected outward along the ground and when this shear is negative, the fluid moves toward the central axis of the vortex (see Serrin[9]). In this last case, two kinds of flows can emerge, according to the sign of the density of stresses $\mathcal{T}$ near the Z-axis: Either the flow is an updraft ($\mathcal{T} > 0$), or when the fluid is descending along the central axis ($\mathcal{T} < 0$), the general motion toward the origin is balanced by a compensating outflow near an intermediate streamcone.

Moreover, we notice that the nonzero effort density $\mathcal{T}$ given by (3.61) leads to an unbounded vertical velocity.[8] This singularity is motivated by the existence of an intense vertical flow inside the core of a tornado caused, for example, by buoyancy forces.[9]

In the same way, our model is likely to integrate intense vertical motions on the axis, in the case of a sink line, as a result of a nonzero vertical force $\mathcal{T}$ given by (3.60).

In our work, the five cases of flows (see below Sec. 3.2) which appear when we add the presence of a source/sink along the central axis are determined by:

- the shear near the ground whose expression is modified because of the contribution of the intensity ξ of the source/sink line, and is given by:

$$\tau_r = \frac{\rho\Gamma^2}{2r^2}(2I_\Omega - P + P_\xi)\,, \tag{3.62}$$

- the intensity of the lateral mass flux characterized by the parameter ξ.

As in the Serrin's model, the sign of the corrected shear (3.62) induces the nature of the flow near the ground. Near the axis, the nature of the flow is determined by the sign of the parameter ξ: In the case ξ positive, streamlines in a meridian plane are outward from the vortex line, whereas in the case ξ negative, streamlines are inward. These remarks allow us to establish different types of flows that the model is able to describe.

3.2. *The flows developed by a swirling source/line vortex*

In agreement with Theorem 3, we resume the results to detail the fluid motions.

The flows modeled by the solutions of problem $(\tilde{\mathcal{P}}_\xi)_2$ *defined in* (2.33) *are characterized by:*

(A) *For a negative shear at the ground* τ_r *and a line of sink* (ξ *negative*), *the function* f *is everywhere negative. The flow possesses only one cell corresponding to the aspiration of the fluid by the vortex/sink axis.*

(B) *For a negative shear at the ground* τ_r *and a line of source* (ξ *positive*), *the function* f *is first negative then positive. The resulting flow is separated in two regions limited by a streamcone that corresponds to the sign change of the function* f. *Inside the cone and near the axis, the fluid is ejected by the source line. On the other hand, near the ground, the fluid moves toward the vortex axis and is ejected outside and along the intermediate streamcone.*

(C) *For a positive shear at the ground* τ_r *and a source line* (ξ *positive*), *the function* f *is everywhere positive. The flow possesses a unique cell and is ejected from the vortex/source axis.*

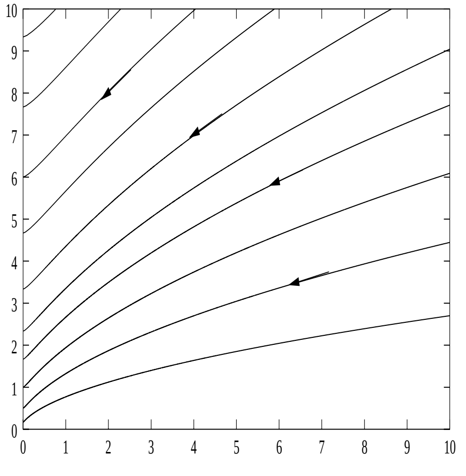
(D) *For a positive shear at the ground τ_r and a sink line (ξ negative), the function f is first positive then negative. The fluid moves toward the sink line along an intermediate streamcone that corresponds to the sign change of the function f. On the other hand, a part of the fluid is ejected along the ground.*

(E) *For certain values of the parameters that correspond to a negative shear τ_r near the ground and a sink line (ξ negative), the function f is first negative, then positive and negative again. This is a very particular flow: One part of the fluid is aspirated by the sink, toward the axis, along an intermediate streamcone that corresponds to the second zero of the function f. Near the ground, the fluid moves toward the axis too. But the totality of the fluid is not driven towards the axis. Therefore, it appears an intermediate streamcone between these two parts of the flow, where the general motion toward the origin is balanced by a compensating outflow along a second streamcone that corresponds to the first zero of the function f.*

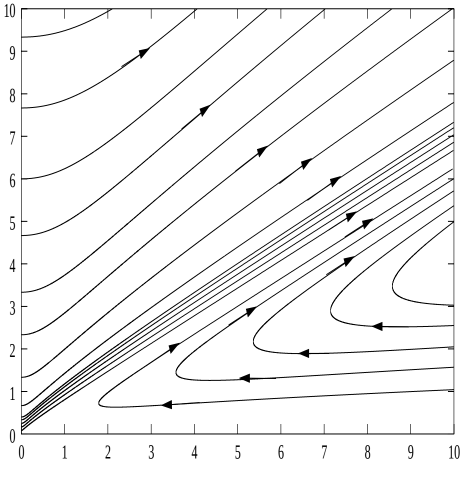
All of these flows are simulated by numerical approximations of the solutions of problem $(\tilde{\mathcal{P}}_\xi)_2$. As usual, a classical way to get a vizualization of these flows is to consider the streamsurfaces, defined[9] by $\Psi(x, R) = RF(x) = C$ and projected on a given meridian plane in Fig. 4.

Remark. One can observe that both cases (A) and (E) correspond to negative values of the shear τ_r near the ground and negative values of ξ, i.e. a central sink line.

One can be surprised that with the same sign of the driving parameters of the flow (τ_r, ξ), which correspond to boundary conditions which are driving the flow toward the central axis, two different kinds of flows emerge.

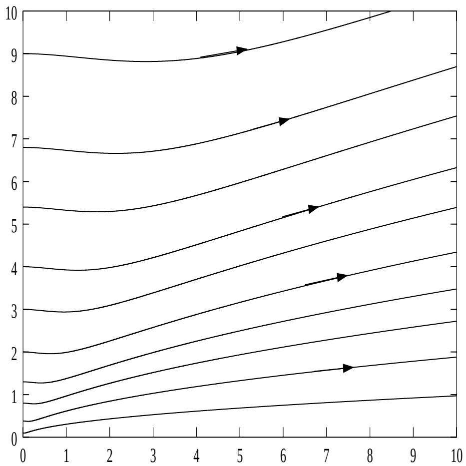


Case A: $\tau_r < 0$ and $\xi < 0$ (zone 1).

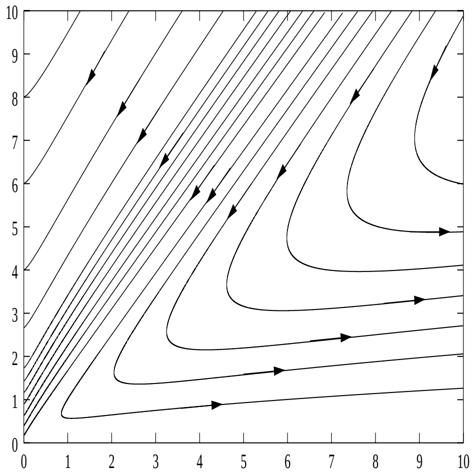


Case B: $\tau_r < 0$ and $\xi > 0$ (zone 2).

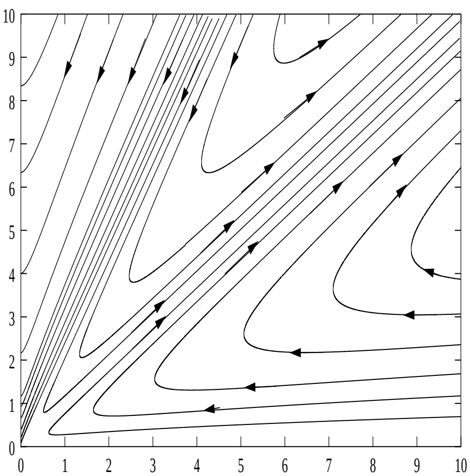
Fig. 4. Numerical simulations and visualization of the flows.



Case C: $\tau_r > 0$ and $\xi > 0$ (zone 3).



Case D: $\tau_r > 0$ and $\xi < 0$ (zone 4).



Case E: $\tau_r < 0$ and $\xi < 0$ (zone 5).

Fig. 4. (*Continued*)

The reason why the case (E) admits an intermediate area with an outflow is due to the order of the magnitude of the intensity of the sink. In case (E), the level of ξ is very small in comparison with its level in case (A).

So, in case (E), neither the sink nor the shear is strong enough to drive the whole flow towards the axis as in the case (A).

Consequently, we observed locally a change of sign of the pressure gradient which allows the possibilty of the appearance of a compensating outflow. This was not obviously remarked in case (A).

3.3. *Analysis of the contribution of the source/sink line*

In the previous sections we exhibited new types of flows, directly connected to the presence of a source/sink line on the swirling vortex axis. In fact, the condition of the presence of a source/sink line leads us to consider either an aspiration or an ejection of mass at the central axis. The flows are slightly different in comparison with those of the Serrin's model. Clearly, the fluid is driven by three main mechanisms which are: the rate of the azimuthal rotation, the vertical shear near the ground (as in the Serrin's model) and the mass flux near the central axis of the vortex. Then in our model, the secondary flow which is superposed on the free vortex has a contribution in terms of energy, which is now in the same order as the energy which is due to the free vortex. Actually, let us observe that the components V_r and V_θ have the same behavior near the vertical axis, whereas we have not determined completely the behavior of the component V_Z. We just showed that it is bounded when ξ is positive. On the other hand, we showed that the presence of the normal component at the central axis (the component V_r), brought a complementary contribution to the central depression induced by the rotation. These behaviors result from the singularity of the solution f of the problem $(\tilde{\mathcal{P}}_\xi)_2$ at the neighborhood of $x = 1$, that is:

$$f \underset{x\to 1^-}{\sim} \frac{\mathcal{R}e\,\xi}{4(1-x)}\,,$$

which is stronger than in the Serrin's modeling, for which the behavior was given by:

$$f \underset{x\to 1^-}{\sim} \frac{\mathcal{R}e^2}{16}(P-1)\ln(1-x)\,.$$

The intensity of the singularity defined by the parameter ξ could be interpreted as a complement of energy to model other phenomena as convection (see Ref. 6).

We have now to precise the limit of our model, particularly concerning the way to take into account the possibility for the velocity to present a nonzero component V_r near the vortex axis. In fact, our model suggests that this contribution is uniform along the central axis. But, it is obvious that this way of description for ejection or absorption of mass by the axis, will not be sufficient enough to describe a swirling vortex with the features of the Minneapolis tornado (see the introduction).

To this end, one would consider a density of a sink or of a source, but not uniformly distributed on the central axis. Therefore, our work contributes to appreciate the type of singularity that will procure a realistic model, that would be completed by an asymptotic analysis to describe the nature of the flow inside the core of a tornado.

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