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A NEW MATHEMATICAL MODEL APPLIED TO TORNADO GENESIS

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Abstract—One of the most important topics in fluid mechanics research is the vortex genesis and the vortex stability. Tornadoes are directly related to these studies: a tornado can be represented by an infinite vortex line in a viscous fluid interacting with a plane boundary surface at right angles to the line. The purpose of this paper is to derive an analytical model which can describe the genesis of these vortex lines in agreement with meteorological mechanisms. This model is built upon a bifurcation process. We consider an axisymmetric non-rotating updraft satisfying an adherence condition at the ground. We show that this flow is unstable and bifurcates to a stationary tornado flow. The startup of a tornado is then modeled.

I. INTRODUCTION

One of the most interesting phenomena that occur in the atmosphere is the tornado. Tornadoes can be described as intense vortices; they are characterized by a core of concentrated vorticity, a life time much longer than their turnover time and adherence to the ground. The tornado dynamics have been investigated by numerous people throughout history. These approaches include analytical or numerical modelling and laboratory experiments in Tornado Vortex Chambers (TVC) (for a review, see Davies-Jones [1]). Theoretical treatments of such vortices are generally limited to axisymmetric flows in basically incompressible fluids. The main difficulty arises from the presence of the boundary surface which leads to a secondary flow due to the forced adherence of the fluid at the surface.

A vortex-type solution of the Navier–Stokes equations has been proposed by Burgers [2] and Rott [3]. The Burger "one-cell" model is a time independent one which was extended to unsteady flows by Rott. But these models do not consider the viscous effects due to the presence of a boundary surface, since only no-stress boundary conditions are satisfied. The theoretical steady model developed by Serrin [4] is a more realistic one. It depends on two scalar parameters P and k (P characterizes the pressure field near the ground and k the rotation rate in the vortex core) and one-cell or two-cell vortices, observed in TVC experiments, can be recovered according to the parameter values. This model can also describe a centrally descending motion. Moreover, it obeys both the strict adherence condition at the boundary surface and the requirement that the velocity approaches zero as one proceeds radially away from the vortex. Recently, Goldshtik and Shtern [5], suggested that tornadoes can also be represented by conical viscous flows including or not a vortex line source. Anyway, all these models only describe stationary tornadoes, especially during their mature phase.

The purpose of the paper is to derive a tornado genesis model based on the Serrin model and a bifurcation analysis. The basic flow is an axisymmetric updraft (or downdraft) with a zero rotation rate satisfying the adherence condition at the boundary surface. For small perturbations with zero rotation rate, we show that the basic flow is stable. But when the rotation rate of the perturbations becomes non zero, the basic flow bifurcates to a Serrin flow and the vorticity generated by the perturbations is focussed.

II. SERRIN MODEL

The Navier–Stokes equations are written in spherical coordinates (R, θ, α), where R is the radial distance from the origin, α is the angle between the radius vector and the z-positive axis, and θ is the meridian angle about the z-axis. The z-axis is then described by $\alpha=0$, the boundary plane $z=0$ by $\alpha=\pi/2$, and the half space $z>0$ by $(R>0,\ 0\leq\alpha<\pi/2)$.

In his model, Serrin [4] considers the following steady velocity field

$$V_R=\frac{F'(x)}{R},\qquad V_\alpha=\frac{F(x)}{r},\qquad V_\theta=\frac{\Omega(x)}{r},\tag{1}$$

where $r=R\sin\alpha$ and $x=\cos\alpha$.

Then the Navier–Stokes equations lead to the integral-differential system

$$f'+f^2=k^2\frac{H_P(x)}{1-x},\qquad \Omega''+2f\Omega'=0,\qquad 0\leq x<1,\tag{2}$$

$$H_P(x)=2\frac{(1-x)}{(1+x)^2}\int_0^x\frac{t\Omega^2}{[1-t^2]^2}\,dt+\frac{2x}{(1-x)(1+x)^2}\int_x^1\frac{\Omega^2}{[1+t]^2}\,dt-\frac{Px}{(1+x)^2},\tag{3}$$

where the function f is defined by $F(x)=2\nu(1-x^2)f(x)$; the parameter k is equal to $1/2\nu$ (ν denotes the kinematic viscosity) and the parameter P characterizes the pressure field near the ground.

The functions f and Ω are subject to the boundary conditions

$$f=\Omega=0\quad\text{when}\quad x=0,\qquad \Omega\to C\quad\text{as}\quad x\to 1^-.\tag{4}$$

The first conditions are prescribed by the adherence property $\mathbf{V}=\mathbf{0}$ when $\alpha=\pi/2$; the last one is derived from the following behaviour: the velocity approaches the vortex value C/r as α tends to zero. Moreover, Serrin [4] has established that the constant C can be supposed equal to one without loss of generality.

III. AN ANALYTICAL MODEL FOR THE TORNADO GENESIS

In the study of the Couette flow between concentric cylinders, Temam [6] develops a bifurcation analysis which can explain the transition between the basic azimuthal flow and the Taylor vortex flow. Following the same ideas, here we first define a basic flow characterized by a zero rotating rate and satisfying the adherence condition. We show that this flow loses its stability when perturbations with non zero rotating rate are applied, and then bifurcates towards a Serrin swirling vortex.

III.1 Basic flow

The basic flow can be either an updraft ($P=P_0>0$) or a downdraft ($P_0<0$). It corresponds to the solutions of the Serrin model with $\Omega(x)=0$ for $x\in[0,1]$.

The system (2), (4) is reduced to

$$f'_{P_0}+f^2_{P_0}=-\frac{k^2P_0x}{(1-x)[1+x]^2},\qquad 0\leq x<1;\qquad f_{P_0}(0)=0.\tag{5}$$

The right-hand-side of equation (5) behaves like

$$\frac{1}{1-x}$$

as x goes to one and leads to a

$$\frac{k^2P_0}{4}\ln(1-x)$$

singular behavior for the solution f_{P_0}. Then we can split f_{P_0} in a singular part and a regular one as follows

$$f_{P_0}(x)=h_0(x)+\mu_0\ln(1-x),$$

with

$$\mu_0 = \frac{P_0 k^2}{4},$$

where h_0 satisfies

$$h_0' + h_0^2 + 2\mu_0 h_0 \ln(1-x) + \mu_0^2 \ln^2(1-x) = \mu_0 \frac{1-x}{(1+x)^2}, \qquad 0 \le x < 1; \qquad h(0) = 0,$$

and belongs to $H_*^1([0, 1]) = \{h \in H^1([0, 1])/h(0) = 0\}$.

This analysis shows that a suitable function Ψ_0 can be defined by:

$$\Psi_0\colon\ H_*^1([0, 1]) \times \mathbf{R} \times \mathbf{R} \to L^2([0, 1]) \times \mathbf{R}$$

$$(h_0, c_0; P_0) \to \left[h_0' + 2c_0 h_0 \ln(1-x) + h_0^2 + c_0^2 \ln^2(1-x) - \mu_0 \frac{1-x}{(1+x)^2}, \qquad \mu_0 - c_0\right].$$

Then $f_{P_0}(x) = h_0(x) + c_0 \ln(1-x)$ is a solution of the equation (5) if and only if $\Psi_0(h_0, c_0; P_0) = 0$. Ψ_0 is a C^1-functional and for each P_0, its partial differential with respect to $X = (h_0, c_0)$ is an isomorphism of $H_*^1 \times \mathbf{R}$ onto $L^2 \times \mathbf{R}$ [7].

The implicit function theorem shows that the equation $\Psi_0(h_0, c_0; P_0) = 0$ locally defines (h_0, c_0) as a function of P_0 in the neighborhood of every point $(\bar{h}_0, \bar{c}_0; \bar{P}_0)$ satisfying $\Psi_0(\bar{h}_0, \bar{c}_0; \bar{P}_0) = 0$. Such points exist when $\bar{P}_0$ and k satisfy the condition $\bar{P}_0 k^2 < \lambda^2 \# 2{,}85^2$ [4]. The existence and the local uniqueness of the solutions of the equation (5) is then proved, P_0 being considered as a parameter. The resulting flow $(f_{P_0}, \Omega = 0)$, where f_{P_0} is solution of (5), is the basic flow for our study. This flow remains stable when small perturbations are applied to P_0, Ω remaining equal to zero; but what happens when the perturbating flow prescribes a non zero rotating rate? We then study this problem when the total perturbated flow is a Serrin swirling vortex.

III.2 Bifurcation analysis

We consider a f_{P_0} basic flow which satisfies (5). By a perturbing process on P_0, this flow becomes a Serrin flow, with a P' value for the parameter P and is defined by the solutions (f, Ω) of (2), (4), with $f(x) = h(x) + f_P(x)$; here f_P is a solution of (5), with P instead of P_0. Furthermore, we assume that Ω satisfies the condition $\Omega^2(1) = P' - P$. This relation has been introduced for mathematical convenience, and its physical meaning will be analysed later.

Then, the functions h and Ω satisfy

$$h'(x) + 2f_P(x)h(x) + h^2(x) = k^2 \frac{H_{\Omega^2(1)}(x)}{1-x},$$

$$\Omega''(x) + 2f_P(x)\Omega'(x) + 2h(x)\Omega'(x) = 0, \qquad 0 \le x < 1, \quad (6)$$

and $h(0) = \Omega(0) = 0$, $\Omega(1) = C$ (C is a given real constant).

Here $H_{\Omega^2(1)}$ is defined by (2) with $\Omega^2(1)$ instead of P.

It is easy to prove that

$$\frac{H_{\Omega^2(1)}}{1-x}$$

behaves like $\Omega(1)\Omega'(1)\ln(1-x)$ as x goes to 1^-.

We define the functional spaces X_1, X_2, X_3, and Y_1, Y_2, Y_3 by

$$X_1 = H_1^*([0, 1]),$$

$$X_2 = H^2([0, 1]) \cap \left(\Omega/\Omega(0) = 0 \quad \text{and} \quad \Omega'(0) = \Omega'(1)\exp\left[2\int_0^1 f_{P_0}(t)\,dt\right]\right)$$

$$X_3 = Y_3 = \mathbf{R}$$

$$Y_1 = Y_2 = L^2(0, 1),$$

and we consider the following functional Ψ

$$\Psi:\quad X_1\times X_2\times X_3\times \mathbf{R}\to Y_1\times Y_2\times Y_3$$
$$(h,\Omega,C;P)\to(h^*,\Omega^*,C^*)$$

with

$$h^*(x)=h'(x)+2h(x)f_P(x)+h^2(x)-k^2\frac{H_{\Omega^2(1)}}{(1-x)},$$
$$\Omega^*(x)=\Omega''(x)+2f_P(x)\Omega'(x)+2h(x)\Omega'(x).$$
$$C^*=\Omega(1)-C$$

Then, $\Psi(h,\Omega,C;P)=0$ describes all the solutions (h,Ω) of the problem (6) which belong to $X_1\times X_2\times X_3$.

In the definition of X_2 the relation

$$\Omega'(0)=\Omega'(1)\exp\left[2\int_0^1 f_{P_0}(t)\,\mathrm{d}t\right]$$

is introduced for mathematical convenience and its physical meaning will be considered later.

Finally, we note that $(0,0,0;P)$ is a trivial solution of the problem (6) for all P values.

The spaces $X_1\times X_2$ and $Y_1\times Y_2$ are Hilbert spaces for the inner products

$$\langle(h_1,\Omega_1),(h_2,\Omega_2)\rangle_{X_1\times X_2}=\int_0^1 h_1'(t)h_2'(t)\,\mathrm{d}t+\int_0^1\Omega_1''(t)\Omega_2''(t)\,\mathrm{d}t+\int_0^1\Omega_1'(t)\Omega_2'(t)\,\mathrm{d}t,$$

$$\langle(u_1,v_1),(u_2,v_2)\rangle_{Y_1\times Y_2}=\int_0^1 u_1(t)u_2(t)\,\mathrm{d}t+\int_0^1 v_1(t)v_2(t)\,\mathrm{d}t.$$

So, the spaces $X_1\times X_2\times X_3$ and $Y_1\times Y_2\times Y_3$ are also Hilbert spaces.

Ψ is a C^2-map and its partial differential $D_x\Psi(0,0,0;P_0)$, with $X=(h,\Omega,C)$, is given by:

$$D_x\Psi(0,0,0;P_0)\cdot(u,\theta,\alpha)=\begin{bmatrix}u'(x)+2f_{P_0}(x)\cdot u(x)\\ \theta''(x)+2f_{P_0}(x)\cdot\theta'(x)\\ \theta(1)-\alpha\end{bmatrix}$$

The bifurcation analysis is based on the following properties of $D_x\Psi(0,0,0;P_0)$, [8]:

Lemma 1. *The kernel of operator $D_x\Psi(0,0,0;P_0)$ is a subspace of $X_1\times X_2\times X_3$; its dimension is equal to one.*

Proof. The kernel of $D_x\Psi(0,0,0;P_0)$ is defined by:

$$\mathrm{Ker}[D_x\Psi(0,0,0;P_0)]=\lambda\left[0,\int_0^x\exp\left(-\int_0^t 2f_{P_0}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t,\int_0^1\exp\left(-\int_0^t 2f_{P_0}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t\right],\qquad \lambda\in\mathbf{R}.$$

Lemma 2. *The orthogonal subspace of the range of $D_x\Psi(0,0,0;P_0)$ is a subspace of $L^2(0,1)\times L^2(0,1)\times\mathbf{R}$; its dimension is equal to one.*

Proof. This result is based on classical lemmas [9], used in the calculus of variations in order to establish Euler's equation, which can be extended to Sobolev spaces and leads to

$$[\mathrm{Im}(D_x\Psi(0,0,0;P_0))]^\perp=\mu\left[0,\int_0^x\exp\left(\int_0^t 2f_{P_0}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t,0\right],\qquad \mu\in\mathbf{R}.$$

Here, the definition of X_2 plays an essential part: $[\mathrm{Im}\,D_x\Psi(0,0,0;P_0)]^\perp$ is not reduced to $(0,0,0)$ since the relation

$$\Omega'(0)=\Omega'(1)\exp\left[2\int_0^1 f_{P_0}(t)\,\mathrm{d}t\right]$$

is satisfied by all the elements of X_2.

LEMMA 3. $L^2(0, 1) \times L^2(0, 1) \times \mathbf{R} = \operatorname{Im} D_x\Psi(0, 0, 0; P_0) \oplus [\operatorname{Im} D_x\Psi(0, 0, 0; P_0)]^{\perp}$.

PROOF. Each point (u, v, λ) of $Y_1 \times Y_2 \times Y_3$ can appear as the sum of one element of $\operatorname{Im} D_x\Psi(0, 0, 0; P_0)$ and one of $[\operatorname{Im} D_x\Psi(0, 0, 0; P_0)]^{\perp}$, and this splitting is unique.

These three lemmas directly lead to the

THEOREM 1. *The operator $D_x\Psi(0, 0, 0; P_0)$ is a Fredholm operator; its index and its codimension are equal to one.*

The final step is based on the Liapunov–Schmidt theorem and the Morse lemma: the Liapunov–Schmidt theorem proves that the solutions $X = (h, \Omega, C)$ of the equation $\Psi(h, \Omega, C; P_0) = 0$ are defined by a bifurcation equation of finite dimension, $F(X, P) \equiv Q\Phi(X + g(X, P); P) = 0$, where X belongs to $\operatorname{Ker} D_x\Psi(0, 0, 0; P_0)$, ($Q$ is the projection on $[\operatorname{Im} D_x\Psi(0, 0, 0; P_0)]^{\perp}$ and g a C^1-functional which is known). Then the Morse lemma can be applied to the functional F:

LEMMA 4. *The point $(0, P_0)$ is a nondegenerate critical point of the functional F.*

PROOF. The Hessian of F at the point $(0, P_0)$ is given by $\det[D^2F(0, P_0)] = -[D^2_{PX}F(0, P_0)]^2$ and $[D^2_{PX}F(0, P_0) \cdot \pi] \cdot (u, v, \alpha) = Q[D^2_{PX}\Psi(0, 0, 0; P_0) \cdot \pi] \cdot (u, v, \alpha)$, where (u, v, α) belongs to $\operatorname{Ker} D_x\Psi(0, 0, 0; P_0)$ and π to $\mathbf{R}$. Then, one can prove that $\det[D^2F(0, P_0)]$ is non zero, [8]. As a consequence, the transversality condition of Hopf theorem is satisfied at the point $(0, 0, 0; P_0)$. The main result can be stated as follows:

THEOREM 2. *The point $(0, 0, 0; P_0)$ is a bifurcation point for the equation $\Psi(h, \Omega; C; P_0) = 0$ and there is exactly one continuous curve of non trivial solution $[(h, \Omega, c)(\varepsilon); P(\varepsilon)]$ bifurcating from $(0, 0, 0; P_0)$.*

III.3 Physical analysis of the model

The results we have established can model the startup of a tornado. More precisely, they can describe the transition between two characteristics flows which are experimentally observed. The first one is the basic flow and is either an updraft or a downdraft with zero rotation rate; this flow exist when the condition $P_0k^2 < \lambda^2 \# 2{,}85^2$ is satisfied and is locally stable with respect to small perturbations of P about P_0. The bifurcation analysis shows that this basic flow bifurcates to a second one which appears to be a Serrin swirling vortex.

The study of the bifurcated flow shows that the total field develops at the neighbourhood of the ground a vertical shear more important than the basic field. This property is implied by the two mathematical relations used in the bifurcation analysis:

$$\Omega^2(1) = P' - P \tag{7}$$

$$\Omega'(0) = \Omega'(1)\exp\left[2\int_0^1 f_{P_0}(t)\,\mathrm{d}t\right], \tag{8}$$

First, if we consider the radial velocity component $V_R^P(\alpha, R)$ of the basic flow defined by a P value, we have

$$\frac{\partial V_R^P(0, R)}{\partial \alpha} = \frac{P}{2\nu R}.$$

Then the relation (7) and some derivations [6] show that the radial velocity component V_R of the P'-Serrin flow satisfies

$$\frac{\partial V_R(0, R)}{\partial \alpha} = \frac{\partial V_R^P(0, R)}{\partial \alpha} + \frac{1}{2\nu R}\left[\Omega^2(1) - 2\int_0^1 \frac{\Omega^2(t)}{(1+t)^2}\,\mathrm{d}t\right]. \tag{9}$$

The bracket term in (9) is strictly positive (the function Ω^2 increases monotonically as x increases from 0 to 1 (Serrin [4]), which implies

$$\frac{\partial V_R(0, R)}{\partial \alpha} > \frac{\partial V_R^P(0, R)}{\partial \alpha}.$$

This first inequality allows a comparison between the vertical velocity shears respectively developed by a P-basic flow and the P'-Serrin flow where P' is defined by (7): the later is stronger than the first one.

Now we consider a P_0-basic flow which bifurcates to a P'-Serrin flow defined by (f_P, h, Ω). It can be proved [8] that the relation (7) implies

$$\int_0^1 f_{P_0}(x)\,\mathrm{d}x < \int_0^1 f_P(x)\,\mathrm{d}x.$$

This last relation yields $P < P_0$ since f_P and f_{P_0} satisfy (5). The following inequalities can also be derived

$$\frac{\partial V_R(0, R)}{\partial \alpha} > \frac{\partial V_R^{P_0}(0, R)}{\partial \alpha} > \frac{\partial V_R^P(0, R)}{\partial \alpha}. \tag{10}$$

The parameter P characterizing the total perturbated flow appears to be strictly smaller than the parameter P_0 defining the basic flow. The bifurcation then is subcritical and the total perturbated flow develops near the ground a vertical velocity shear stronger than the vertical velocity shear characterizing the initial basic flow. Moreover, the intensity of the vertical velocity shear near the ground for P or P_0-basic flows and the P'-Serrin flow can be compared by using the inequalities (10).

IV. CONCLUSION

The bifurcation process which has been established can be physically analysed as follows: the basic flow is generated simultaneously by a strong vertical gradient of temperature and by a storm in the troposphere: it is a non rotating updraft. Then we consider a horizontal perturbation which develops a high vertical velocity shear generating horizontal vorticity. The interaction of these two flows results in the genesis of a tornado. This picture is in good agreement with the phenomenological mechanisms of tornado genesis which are usually suggested [1]. More precisely, the bifurcation theorem guarantees that the initial non rotating flow bifurcates to a strongly rotating steady state, which is modeled by the Serrin solutions. Therefore, the final flow presents a vertical velocity shear near the ground which is stronger than the initial one.

Further developments about the tornado genesis have to take into account the thermal characteristics of the atmosphere. In fact, our model can explain a tornado genesis but does not define some forecasting criterion. For this purpose it will be helpful to include the thermodynamic aspects in the process.

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