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Bifurcation theory applied to buckling states of a cylindrical shell

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1. Introduction

The purpose of this work is to develop a new approach to exhibit nontrivial solutions, buckled states, of nonlinear equilibrium equation for inextensible elastic tubes. In the reference configuration, each circular cross section of the tube is subjected to a uniform external pressure. The main tool for this analysis is the bifurcation theory and more precisely the Liapunov-Schmidt decomposition and the bifurcation theorem for simple multiplicity.

The nonlinear problem with which we deal is based upon the Euler-Bernoulii theory of the elastica. It was first considered by Levy [1] who reduced it to the investigation of an algebraic problem involving elliptic integrals. Later, Carrier [2] reconsidered the problem, and obtained an approximate solution to the algebraic formulation for small displacements, and discussed some other related problems. Then, Tadjbakhsh and Odeh [3] used an asymptotic solution of nonlinear differential equations to show the existence of buckled states.

The first part of our paper describes the mathematical model we obtain under Euler-Bernoulli hypothesis. Precisely, we establish the nonlinear integro-differential equation subjected to the boundary conditions for our functional analysis. We must deal with a family of one parameter λ solutions. The second part is devoted to the bifurcation analysis from which one can prove that a nontrivial branch of solutions occurs for adequate values of the parameter λ. We then conclude with physical interpretations about the degenerated states of the cylindrical shell which can appear if the shell is submitted to the critical pressures described before. Furthermore, these states are plotted on the bifurcation diagram which shows the dependence between the cross-sectional area and the pressure.

Practical applications of this phenomenon can be found in the study of veins, bronchii, and many other vessels in the human body which are flexible enough to be capable of collapse if subjected to suitable applied external and internal loads [4].

2. Formulation of the problem

We considered the equilibrium configurations of an elastic tube subjected to a uniform external pressure p. The pressure load is assumed to be constant in magnitude and pointed in the direction of the inward normal vector on the deformed tube. The tube is considered to be a thin, elastic, cylindrical shell, of axis $\boldsymbol{x}$, with a circular cross section in the undeformed shape. In addition, it is assumed to be homogeneous, of constant thickness and infinite length, and to have an inextensible average surface [5].

In this case, the formulation of the problem is two dimensional and each cross section is described by a line, called mean line Γ, equidistant from the outer and inner boundaries of the section. In the reference configuration, the Γ oriented curve is a circle with radius R and center of coordinates $(0, R)$ in a coordinate system $\mathscr{R}(O, y, z)$.

The curvilinear abscissa s of a point M, of coordinates (y, z) in $\mathscr{R}$, is defined on Γ. Each point M is associated with a local coordinate system $(M, \boldsymbol{t}, \boldsymbol{n})$ where $\boldsymbol{t}$ is the vector tangent to Γ at M and such that $\boldsymbol{n} = \boldsymbol{t} \wedge \boldsymbol{x}$. The angle between the axis y and the tangent to Γ at M is noted θ (Fig. 1).

The coordinates $y(s)$ and $z(s)$ satisfy the relations:

$$\frac{dy}{ds}(s) = \cos\theta(s), \qquad \frac{dz}{ds}(s) = \sin\theta(s), \qquad \forall s \in [-\pi, \pi].$$

We assume that these coordinates verify the homogeneous conditions corresponding to neither displacements nor rotation at points $s = -\pi$; this is to eliminate all the rigid body displacements:

$$\theta(-\pi) = 0, \qquad y(-\pi) = 0, \qquad z(-\pi) = 0. \tag{1}$$

Furthermore, since the deformed tube is a closed shell, the following conditions must hold:

$$\theta(s + 2\pi) = \theta(s) + 2\pi, \qquad y(s + 2\pi) = y(s), \qquad z(s + 2\pi) = z(s). \tag{2}$$

Internal efforts for each unit length at a point M of Γ are the resulting efforts $\mathscr{T} = N\boldsymbol{t} + Q\boldsymbol{n}$ of the traction or compression force $N(s)$, the shearing force $Q(s)$, and the bending moment $\mathscr{M}(s)\boldsymbol{x}$. The outside forces per unit length are given by $\mathscr{T}_F = -p\boldsymbol{n}$.

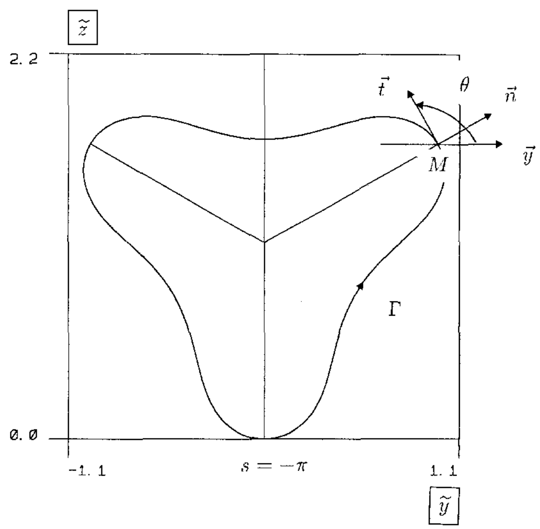

Figure 1
Example of the cross-sectional area deformation of a thick tube. Each point M is associated to a local coordinate system $(M, \boldsymbol{t}, \boldsymbol{n})$.

Outside the points on which concentrated forces are exerted, the equilibrium equations projected on the local coordinate system read:

$$\frac{dN}{ds} + Q\frac{d\theta}{ds} = 0, \tag{3}$$

$$\frac{dQ}{ds} - N\frac{d\theta}{ds} - p = 0, \tag{4}$$

$$\frac{d\mathcal{M}}{ds} - Q = 0. \tag{5}$$

On the other hand, if the deformations are considered to be small, the behaviour law is given by:

$$\mathcal{M} = D\left(\frac{d\theta}{ds} - \frac{1}{R}\right) \tag{6}$$

where $D = \{Eh_0^3\}/\{12(1-\nu^2)\}$ is the flexural rigidity modulus, E is Young's modulus, ν is Poisson's ratio, h_0 is the wall thickness.

This law is valid only if $h_0(d\theta/ds) \ll 1$, and in this case, the displacements of the wall are often large.

The main dimensionless variables are given as follows:

$$\tilde{s} = \frac{s}{R}, \qquad \tilde{h}_0 = \frac{h_0}{R}, \qquad \tilde{k} = R\frac{d\theta}{ds}, \qquad \tilde{p} = \frac{pR^3}{D}.$$

Combining equations (3)–(6) and using dimensionless variables, one finds for $\tilde{s} \in]-\pi, \pi[$:

$$\frac{d^2\tilde{k}}{d\tilde{s}^2} + \frac{1}{2}\tilde{k}^3 - c\tilde{k} - \tilde{p} = 0 \tag{7}$$

where c is an integration constant.

Moreover, if we suppose that shearing forces $Q(s)$ are continuous at the fixing point, recalling that the average surface is inextensible, we have:

$$\frac{d\tilde{k}}{d\tilde{s}}(-\pi) = \frac{d\tilde{k}}{d\tilde{s}}(\pi) = 0, \qquad \int_{-\pi}^{\pi} \tilde{k}(s)\, ds = 2\pi. \tag{8}$$

Because of the symmetric position of the fixing point, the solution $\tilde{k}$ of equations (7) and (8) can be searched as an even function.

Now, integrating equation (7) on $[-\pi, \pi]$ and using the conditions (8) yield the following relation:

$$c = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\tilde{k}^3 - \tilde{p}\right) ds = \frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{2}\tilde{k}^3 - \tilde{p}\right) ds.$$

Lastly, if we introduce v such that $v = \tilde{k} - 1$, the final formulation of the problem is:

$$(\boldsymbol{P})\begin{cases} v'' + \dfrac{1}{2}v^2(v+3) + (1+\tilde{p})v - \dfrac{(1+v)}{2\pi}\displaystyle\int_0^{\pi} v^2(v+3)\, ds = 0, & \text{on }]0, \pi[\\ v'(0) = v'(\pi) = 0, & \\ v(-s) = v(s), \quad \forall s \in [0, \pi] & \end{cases}$$

where v' denotes dv/ds.

3. Bifurcation analysis

This section is devoted to the nonuniqueness of the problem solutions (***P***). More precisely, we study if there exists critical values of the parameter $\tilde{p} > 0$, generating nontrivial solutions of the problem (***P***). For this purpose, we introduce the functional ϕ defined by:

$$\phi: V \times \mathbb{R} \to W$$

$$(v, \lambda) \to \bar{v} \equiv v'' + \frac{1}{2} v^2(v+3) + \lambda v - \frac{(1+v)}{2\pi} \int_0^{\pi} v^2(v+3)\, ds,$$

where $\lambda = \tilde{p} + 1$ and

$$V = H^2(-\pi, \pi) \cap \{v / v \text{ is even}, v'(0) = v'(\pi) = 0\},$$

$$W = L^2(-\pi, \pi) \cap \{\bar{v} / \bar{v} \text{ is even}\}.$$

Then, $\phi(v, \lambda) = 0$ describes all the solutions of the problem (***P***), for $v \in V$.

Remarks:

(i) the spaces $H^2(-\pi, \pi)$ and $L^2(-\pi, \pi)$ are respectively the classical Sobolev and Hilbert spaces.

(ii) $(0, \lambda)$ is the trivial branch of $\phi \equiv 0$, for all real numbers λ. Therefore our interest lies in finding $(0, \lambda_0)$ from which a nontrivial branch occurs.

The functional frame presented before is convenient for our bifurcation analysis. More precisely, the properties of V and W are detailed in

Lemma 1: The space V (resp. W) as a closed subset of $H^2(-\pi, \pi)$ (resp. $L^2(-\pi, \pi)$) is a Hilbert space.

4. Properties of the functional ϕ

In this section we characterize the operator $D_v\phi(0, \lambda_0)$ as a non invertible one, for suitable values of the parameter λ_0. This property is the necessary condition which must be verified by $D_v\phi(0, \lambda_0)$ to insure that $(0, \lambda_0)$ does not satisfy the condition of the implicit function theorem.

First of all, let us remark that the functional ϕ is a C^2-map and its partial differential $D_v\phi(0, \lambda_0)$ is given by:

$$D_v\phi(0, \lambda_0) \cdot w \equiv \bar{w} = w'' + \lambda_0 w.$$

We give below the suitable conditions on λ_0 such that the kernel of $D_v\phi(0, \lambda_0)$ is not reduced to zero.

Lemma 2: For each $\lambda_0 = n_0^2$, given $n_0 \in \mathbb{N}$, the kernel of $D_v\phi(0, \lambda_0)$ is a one dimensional subset of V.

Proof: The kernel of $D_v\phi(0, \lambda_0)$, with $\lambda_0 = n_0^2$, is given by:

$$\operatorname{Ker}\{D_v\phi(0, \lambda_0)\} = \{w \in V / w(s) = A \cos(n_0 s), A \in \mathbb{R}\}.$$

The condition $\lambda_0 = n_0^2$ is sufficient to achieve the characterization of $D_v\phi(0, \lambda_0)$.

We note that for any other values of λ_0, the kernel of $D_v\phi(0, \lambda_0)$ is reduced to zero.

Lemma 3: The range of $D_v\phi(0, \lambda_0)$, where $\lambda_0 = n_0^2$ with $n_0 \in \mathbb{N}$, is closed into W.

The final step of this paragraph concerns the codimension of the operator $D_v\phi(0, \lambda_0)$. Precisely, we have the

Lemma 4: The orthogonal subspace of the range of $D_v\phi(0, \lambda_0)$, where $\lambda_0 = n_0^2$ with $n_0 \in \mathbb{N}$, is a one dimensional subset of W. Moreover, we have $[\mathrm{Im}\{D_v\phi(0, \lambda_0)\}]^\perp = \mathrm{Ker}\{D_v\phi(0, \lambda_0)\}$.

The combination of these lemmas directly leads to the

Theorem 1: The operator $D_v\phi(0, \lambda_0)$, with $\lambda_0 = n_0^2$ for a given integer n_0, is a Fredholm operator; its index and its codimension are equal to one.

The main consequence of this theorem is the application of the Liapunov-Schmidt decomposition [7] or [8], which reduces the bifurcation problem to a finite dimensional one. Here, the reduction leads us to the case of simple multiplicity, for which the bifurcation arises at the point $(0, \lambda_0)$ if the functional ϕ satisfies the transversality condition [7] or [8]. This is the proposal expressed by the following theorem:

Theorem 2: For all λ_0 given as the square of an integer, the point $(0, \lambda_0)$ is a bifurcation point for the equation $\phi(v, \lambda) = 0$ and there is exactly one continuous curve of nontrivial solutions $(v(\varepsilon), \lambda(\varepsilon))$ bifurcating from $(0, \lambda_0)$.

Proof: We just have to establish the transversality condition.

Let $X = \{\bar{g} \in W / \exists(\delta\lambda, g) \in \mathbb{R} \times \mathrm{Ker}\{D_v\phi(0, \lambda_0)\}: [D^2_{\lambda v}\phi(0, \lambda_0) \cdot \delta\lambda] \cdot g = \bar{g}\}$,

this space is usually denoted in the literature $D^2_{\lambda v}\phi(0, \lambda_0)g$. We claim that:

$X \cap \mathrm{Im}\{D_v\phi(0, \lambda_0)\} = \{0\}$.

The operator $D^2_{\lambda v}\phi(0, \lambda_0)$ is given by:

$[D^2_{\lambda v}\phi(0, \lambda_0) \cdot \delta\lambda] \cdot g \equiv \bar{g} = \delta\lambda g, \qquad \forall(\delta\lambda, g) \in \mathbb{R} \times V.$

For $g \in \mathrm{Ker}\{D_v\phi(0, \lambda_0)\}$, $\bar{g} \in \mathrm{Ker}\{D_v\phi(0, \lambda_0)\} \equiv [\mathrm{Im}\{D_v\phi(0, \lambda_0)\}]^\perp$.

Furthermore, if $\bar{g}$ belongs to $\mathrm{Im}\{D_v\phi(0, \lambda_0)\}$ too, then $\bar{g} \equiv 0$ because of the splitting of W. This achieves the proof of the theorem.

5. Physical interpretation

Theorem 2 shows that a nontrivial branch of solutions arises for each $\lambda_0 = n_0^2$, $n_0 \in \mathbb{N}$. Recalling that $\lambda_0 = \tilde{p}_0 + 1$ gives the bifurcation values of the dimensionless pressure $\tilde{p}_0 = n_0^2 - 1$. This result was already found in the past by several authors like Timoshenko and Gere [9], Courbon [10], Tadjbakhsh and Odeh [3]. The approach of the first two authors is based on the minimization of energy. This method uses the Fourier series development of the solution. Tadjbakhsh and Odeh [3] found the critical values of the pressure by computation of power series solutions in an amplitude ε.

The limit of these two previous methods is that there is no way to conclude about the number of branches which arise at the critical values of the parameter $\tilde{p}$. Precisely, Theorem 2 describes the nature of the bifurcation. Indeed, there is one and only one branch of nontrivial solutions at the bifurcation point.

Now, let us give physical interpretation of our result. One can observe that dimensional pressures from which the nature of deformations is changing, are given by:

$$p = (n^2 - 1)\frac{D}{R^3}.$$

Then, we remark that the case $n = 1$ has to be left out because we only consider strictly positive values of the pressure. So, the first bifurcation occurs for $n = 2$, which yields the bifurcation pressure:

$$p = \frac{3D}{R^3}.$$

Moreover, each value of n corresponds to a deformed tube which has n axes of symmetry [3]. Comparing this result with Theorem 2 gives a new information; only two shapes of the tube are observable at the bifurcation point: the tube must either have a circular cross section or n axes of symmetry.

Finally, we present one last interpretation about the variations of the cross-sectional area as a function of the pressure. The cross-sectional area $\tilde{A}$, with the dimensionless variables, is given by:

$$\tilde{A} = 2\int_0^{\pi}\left(\int_0^{s}\cos\theta(\tau)\,d\tau\right)\sin\theta(s)\,ds. \tag{10}$$

A normalised area $\bar{A} = \tilde{A}/\tilde{A}_0$ is defined, where $\tilde{A}_0$ is the initial area.

The section area $\bar{A}$ is calculated by numerical quadrature using equation (10). Figure 2 shows the bifurcation diagram of which one can observe that bifurcations occur at the theoretical values exhibited in Theorem 2.

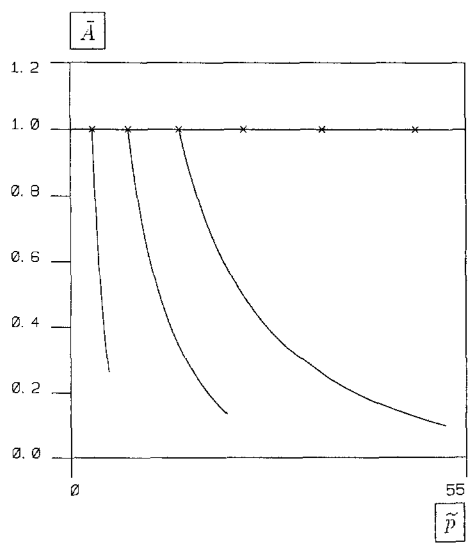

Figure 2
Pressure variation as a function of the cross-sectional area. Example of nontrivial branches calculated for $n = 2$, 3 and 4. The cross symbol represents critical values.

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Abstract

Veins, bronchii, and many other vessels in the human body are flexible enough to be capable of collapse if submitted to suitable applied external and internal loads. One way to describe this phenomenon is to consider an inextensible elastic and infinite tube, with a circular cross section in the reference configuration, subjected to a uniform external pressure. In this paper, we establish that the nonlinear equilibrium equation for this model has nontrivial solutions which appear for critical values of the pressure. To this end, the tools we use are the Liapunov-Schmidt decomposition and the bifurcation theorem for simple multiplicity. We conclude with the bifurcation diagram, showing the dependence between the cross-sectional area and the pressure.

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