New mathematical models for media exposure

J. Chaskalovic *
Department of Computer Sciences and Mathematics
Ariel University Center of Samaria
40700 Ariel
Israel
and
Institute Jean le Rond d'Alembert
University Pierre and Marie Curie – Paris VI
4 Place Jussieu 75252
Paris Cedex 05
France

Abstract

Important features which are considered in media research are media efficiency and return on investments. To appreciate the powerful level of several channels of communication, one has to firstly consider the basic measure of the impact for a given advertising campaign, that is to say, the individual distribution of exposure and therefore the estimation of the global exposure of a given target. Having pointed out the known models, we will show how one can release part of hypotheses, under which these models are founded to describe new behaviors of media exposure. This will be particularly the case by introducing the Markov chains to describe an unsteady dependency between several issues or TV spots regarding a given advertising campaign. As a consequence we will show that the new models estimate more precisely the net cover of a given campaign composed by N advertising spots.

Keywords and phrases : *Markov chains, Volterra integro equation, mathematical model, media, individual frequency exposure, cover.*

1. Introduction

Research activity in Media context must bring to the professionals working either in media agencies or in marketing department for a given company, the relevant tools to estimate the return on investments due to the advertising.

* *Present address*: Ariel University Center of Samaria, Ariel, Israel.

Journal of Interdisciplinary Mathematics
Vol. 12 (2009), No. 3, pp. 395–408

One of the main questions consists on the calcu-lation of the exposure frequency distribution which determines for a set of N advertising issues which belong to a group of magazines for example, (either N spots in television or in radio, etc.), and also the average probability that a given population is exposed k times, $(k = 1, N)$.

To this end, numerous models based on a corpus of hypotheses were developed. These hypotheses entirely determine the capacity to model, and therefore, to apprehend a limited class of individual media behaviors.

As a consequence, famous indicators are estimated. The net cover of a given advertising campaign composed by N issues (the percent of the individuals in a given target of a population which is exposed at least once) and the repetition which qualifies the average number of times that a given individual is exposed during the advertising campaign are the most useful and famous ones.

Several authors proposed different methods to estimate these indicators. Mainly, one can find in the literature two kinds of models based on mathematical tools which basically are either determinist, (as Agostini [1], Politz cited by [11] and Morgensztern [15]; for a review see [16]), or probabilist.

Having pointed out the known probabilistic models, we will show how one can release part of hypotheses, under which these models are founded, to describe new behaviors of media exposure.

Accordingly, we will show that the new models more precisely estimate the net cover of a given campaign composed by N issues.

To this end, we will consider the cover R_N which corresponds to the percent of people which are exposed at least once during the advertising campaign composed by N spots and particularly the asymptotic cover R_∞ as the maximum threshold of the cover when the number of commercials N goes to infinity.

To simplify our purpose, in this paper, we will consider advertising campaigns only composed by a media support, let say, for example, television.

Therefore, a campaign is made by N TV spots S_n, $(n = 1, N)$ and we will describe an advertising campaign as follows:

S_1	S_2	$\cdots$	S_N

2. A first generation of models: the aggregation models

We consider a given target in the population composed by I homogeneous classes of people. It means that each class is characterized by a set of specific features.

We introduce the two quantities $e^n_{i,N}$ and E^n_N as follows.

Definition 1. Let a given individual being in the class i, $(i = 1, I)$. We denote by $e^n_{i,N}$ the probability such that the individual is exactly exposed n times on N given TV spots.

Definition 2. We define the quantities E^n_N as the average probability that the total population of a target is exactly exposed n times on the same N TV spots.

Therefore, the aggregative models consist to determine E^n_N in relation with $e^n_{i,N}$, along the following formula:

$$E^n_N = \sum_{i=1}^{I} \pi_i e^n_{i,N} , \qquad \text{where } \sum_{i=1}^{I} \pi_i = 1 . \tag{1}$$

One can observe that in (1), the weights π_i describe the percent of people of the target which corresponds to particular features represented by the class of individuals i.

2.1 *The Full binomial law*

We begin our process to model the average probability E^n_N by the individual exposure distribution $e^n_{i,N}$ when considering the following hypothesis:

(1) The individual exposure probability $p^{(n)}_i$ to the different TV spots is stationary:

$$\forall\, n = 1, N : p^{(n)}_i \equiv p_i. \tag{2}$$

In other words, the TV campaign has the next structure:

s_1	s_j	$\cdots$	s_N
p_i	p_i	$\cdots$	p_i

(2) The individual exposure to a given spot S_n is independent to any previous exposure to another spot S_k, $(k \leq n)$.

Then, the individual frequency exposure distribution $e_{i,N}^n$ follows the binomial law [10]:

$$e_{i,N}^n = C_N^n p_i^n (1 - p_i)^{N-n} , \tag{3}$$

where C_N^n denotes the binomial coefficient:

$$C_N^n \equiv \frac{N!}{(N-n)!n!} . \tag{4}$$

As an immediate consequence, the corresponding aggregative model called the full binomial law can be written as:

$$E_N^n = \sum_{i=1}^{I} \pi_i e_{i,N}^n = \sum_{i=1}^{I} \pi_i [C_N^n p_i^n (1 - p_i)^{N-n}] . \tag{5}$$

As we explained in the introduction, we will proceed to the determination of the cover R_N and therefore to R_∞, when N goes to infinity, as an indicator of comparison between the different models we will exhibit.

Lemma 1. *The asymptotic cover R_∞ of the full binomial law is given by:*

$$R_\infty = 1 - \sum_{p_i=0} \pi_i . \tag{6}$$

Proof. Because the definition of R_N, we can write:

$$R_N = 1 - E_N^0 = 1 - \sum_{i=1}^{I} \pi_i (1 - p_i)^N. \tag{7}$$

In expression (7) we separate the individuals which have a none zero probability of exposure to those called the zero segment or the absolute non viewers which correspond to the condition $p_i = 0$:

$$R_N = 1 - \sum_{p_i=0} \pi_i - \sum_{p_i \neq 0} \pi_i (1 - p_i)^N. \tag{8}$$

Then, from equation (8), we easily obtain the asymptotic cover R_∞ given by (6).

In other terms, the maximum threshold R_∞ of the cover R_N is reached when we withdraw to the whole target the individuals which belong to the zero segment. □

2.2 *The full steady Markov binomial law*

In this section we are interesting to release part of the hypothesis which led to the previous models.

Especially, we will take into account a first level of conditional exposure between two consecutive TV spots S_{n-1} and S_n, (see [7] and [8] for application in press advertising).

For this purpose, we firstly use the steady and homogeneous Markov chains, (see for example [14]), introduced by Andrei Andreevich Markov (1856-1922) as follows.

Let us denote $P_i^{(n)}$ the individual probability vector of exposure defined by:

$$P_i^{(n)} = \begin{pmatrix} \text{Prob}\{X_i^{(n)} = 1\} \\ \text{Prob}\{X_i^{(n)} = 0\} \end{pmatrix} = \begin{pmatrix} p_i^{(n)} \\ q_i^{(n)} \end{pmatrix}. \tag{9}$$

Then, the conditional behavior between S_{n-1} and S_n is modeled by:

$$\forall\, n = 1, N : P_i^{(n)} = \mathrm{T}_i P_i^{(n-1)}, \tag{10}$$

where the transition matrix T_i, does not depend on n because our hypothesis of homogeneous process.

On the other hand, in this section we only consider *steady state* Markov chains, so:

$$P_i = \mathrm{T}_i P_i\,, \tag{11}$$

with: $P_i = \begin{pmatrix} p_i \\ q_i \end{pmatrix}$.

We denote by a_i and b_i the two conditional individual probabilities as:

$$a_i \equiv \text{Prob}\{X_i^{(n)} = 0 \setminus X_i^{(n-1)} = 1\}, \tag{12}$$

$$b_i \equiv \text{Prob}\{X_i^{(n)} = 1 \setminus X_i^{(n-1)} = 0\}. \tag{13}$$

As an immediate consequence of the definition the conditional probabilities, one must deal with:

$$T_i^{(11)} + T_i^{(21)} = 1, \tag{14}$$

$$T_i^{(12)} + T_i^{(22)} = 1. \tag{15}$$

Then, the transition matrix of individual conditional probabilities T_i can be written along the following structure:

$$\mathrm{T}_i = \begin{pmatrix} 1 - a_i & b_i \\ a_i & 1 - b_i \end{pmatrix}. \tag{16}$$

Relation (16) implies the following relations between a_i, b_i p_i and q_i:

$$p_i = \frac{b_i}{a_i + b_i}\,, \qquad q_i = \frac{a_i}{a_i + b_i}\,. \tag{17}$$

Regarding of the dependency between two consecutive spots S_{n-1} and S_n we choose to model by Markov chains (16), the individual distribution of exposure frequencies cannot be disentangled, except for the term E_N^0 which permits to obtain the cover of the complement.

It's the reason why we only put forward the probability of a sequence of exposure/non exposure to N TV spots.

Before, let us introduce Bernoulli's random variable as follows:

Definition 3. Bernoulli's random variable $X_i^{(j)}$ which qualifies the individual exposure is equal to 1 if the individual i is exposed to the spot S_j and 0 if not.

Therefore, we have the following result.

Theorem 1. *Regarding to the individual behavior of exposition modeled by steady Markov process* (16), *the probability of a sequence of exposure/non exposure to N TV spots is given by:*

Let $x_n, (n = 1, N)$ be an integer, such that $x_n \in \{0, 1\}$. Then:

$$\text{Prob}\{X_i^{(1)} = x_1, \ldots, X_i^{(N)} = x_N\} = p_i a_i^{\alpha} b_i^{\beta} (1 - b_i)^{\gamma} (1 - a_i)^{\delta}, \tag{18}$$

where:

$$\alpha = S - R - x_N\,, \tag{19}$$

$$\beta = S - R - x_1\,, \tag{20}$$

$$\gamma = R\,, \tag{21}$$

$$\delta = R - 2S + x_1 + x_N + N - 1 \tag{22}$$

and

$$S = \sum_{n=1}^{N} x_n \quad \textit{and} \quad R = \sum_{n=2}^{N} x_{n-1} x_n\,. \tag{23}$$

Proof. One has:

$$\text{Prob}\{X_i^{(1)} = x_1, \ldots, X_i^{(N)} = x_N\}$$
$$= \text{Prob}\{X_i^{(N)} = x_N / X_i^{(N-1)} = x_{N-1}\} \cdot \text{Prob}\{X_i^{(1)} = x_1, \ldots, X_i^{(N-1)} = x_{N-1}\}. \tag{24}$$

Then:

$$\begin{aligned}&\mathrm{Prob}\{X_i^{(1)} = x_1, \ldots, X_i^{(N)} = x_N\}\\&= \mathrm{Prob}\{X_i^{(1)} = x_1\} \prod_{n=2}^{N} \cdot \mathrm{Prob}\{X_i^{(n)} = x_n / x_i^{(n-1)} = x_{n-1}\}.\end{aligned} \tag{25}$$

The hypothesis of homogeneity of Markov processes on the one hand, and the particular structure of the transition matrix T_i, on the other hand, allows us transform the conditional probabilities appearing in the expression (30) and write:

$$\begin{aligned}&\mathrm{Prob}\{X_i^{(n)} = x_n / X_i^{(n-1)} = x_{n-1}\}\\&= (1-a_i)^{x_{n-1}x_n} b_i^{(1-x_{n-1})x_n} a_i^{x_{n-1}(1-x_n)} (1-b_i)^{(1-x_{n-1})(1-x_n)}.\end{aligned} \tag{26}$$

Then, replacing (31) into (30) and after some elementary calculations, one get the desired result. □

We are now in position to give the evaluation of the asymptotic cover R_∞ of the full steady Markov binomial law.

Lemma 3. *The asymptotic cover R_∞ of the full steady Markov binomial is given by:*

$$R_\infty = 1 - \sum_{b_i=0, q_i=1} \pi_i - \sum_{b_i=0, q_i \neq 1} \pi_i g_i \,. \tag{27}$$

Proof. Although the expression (23) is quite inconvenient in practice, the probability of exposure to all or to none of the N TV spots may be calculate:

$$\mathrm{Prob}\{X_i^{(1)} = 1, \ldots, X_i^{(N)} = 1\} = p_i(1-a_i)^{N-1}, \tag{28}$$

$$\mathrm{Prob}\{X_i^{(1)} = 0, \ldots, X_i^{(N)} = 0\} = q_i(1-b_i)^{N-1}. \tag{29}$$

Particularly, formula (34) permits us to calculate the corresponding cover R_N for N TV spots.

Indeed, we have:

$$R_N = 1 - \sum_{i=1}^{I} \pi_i \, \mathrm{Prob}\{X_i^{(1)} = 0, \ldots, X_i^{(N)} = 0\} \tag{30}$$

$$= 1 - \sum_{i=1}^{I} \pi_i q_i (1-b_i)^{N-1} \tag{31}$$

$$= 1 - \sum_{b_i=0, q_i=1} \pi_i - \sum_{b_i=0, q_i \neq 1} \pi_i q_i - \sum_{b_i \neq 0} \pi_i q_i (1-b_i)^{N-1}. \tag{32}$$

As a consequence, one obtains the asymptotic cover R_∞ given by (32) when N goes to infinity. □

Remark. Another characterization can be done concerning people which have the features $b_i = 0$ and $q_i \neq 1$.

From relations (22), we observe that individuals which satisfy the two above conditions correspond to those where $a_i = 0$.

In other words, there are a subgroup of people such that a_i and b_i are equal to 0.

Those individuals are called the *ultra loyal* people. Regarding the definitions of a_i and b_i, they correspond to those such that one can be "sure" that if they were exposed to the spot S_{n-1} they will be exposed to the spot S_n, and if they were not exposed to the spot S_{n-1} they will not be exposed to the spot S_n too.

It's the reason why we keep in mind that the asymptotic cover R_∞ of the full steady Markov binomial is equal to:

$$R_\infty = 1 - \sum_{b_i=0, q_i=1} \pi_i - \sum_{a_i=b_i=0, q_i \neq 1} \pi_i q_i . \tag{33}$$

3. A new generation of models

3.1 *Unsteady and conditional individual probability of exposure: the full unsteady Markov binomial law*

We now proceed to the extension of the full steady Markov binomial exhibited in the previous section by the unsteady one which was introduced for the first time by the author in 1995, [6].

More precisely, we do authorize for each individual to change his behavior in terms of exposure from a given spot to another one.

That is to say that the situation we would like to describe is the following one:

s_1	s_j	$\cdots$	s_N
p_i^1	p_i^2	$\cdots$	p_i^N

Another time, we choose to model the correlation between two consecutive spots by the help of the homogeneous Markov process (15).

Therefore, one can write:

$$\forall\, n = 1, N : P_i^{(n)} = \left(\mathrm{T}_i\right)^{n-1} P_i^{(1)}, \tag{34}$$

which is equal to the developed formulation:

$$\begin{pmatrix} p_i^{(n)} \\ q_i^{(n)} \end{pmatrix} = \begin{pmatrix} 1-a_i & b_i \\ a_i & 1-b_i \end{pmatrix}^{(n-1)} \begin{pmatrix} p_i^{(1)} \\ q_i^{(1)} \end{pmatrix} \tag{35}$$

$$= \begin{pmatrix} [1-a_i-b_i]^{n-1} p_i^{(1)} + (1-[1-a_i-b_i]^{n-1}) \frac{b_i}{a_i+b_i} \\ [1-a_i-b_i]^{n-1} q_i^{(1)} + (1-[1-a_i-b_i]^{n-1}) \frac{a_i}{a_i+b_i} \end{pmatrix}. \tag{36}$$

Depending on the sign of the two conditional probabilities a_i and b_i, relations (41) describe three kinds of individual behaviors of exposure [6].

Theorem 2. *Let $p_i^{(n)}$ be the sequence of individual probabilities of exposure defined by* (41). *Then, one must deal with the following media behaviors:*

(1) *In any case of parameters a_i and b_i the sequence of probabilities $p_i^{(n)}$ has a limit $p_i^{(\infty)}$ corresponding to:*

$$p_i^{(\infty)} = \frac{b_i}{a_i + b_i}\,. \tag{37}$$

(2) *If $0 < a_i + b_i < 1$ and $p_i^{(1)} > \frac{b_i}{a_i+b_i}$, then the sequence of probabilities $p_i^{(n)}$ is decreasing until the limit threshold $p_i^{(\infty)}$ defined by* (42).

(3) *If $0 < a_i + b_i < 1$ and $p_i^{(1)} < \frac{b_i}{a_i+b_i}$, then the sequence of probabilities $p_i^{(n)}$ is increasing until $p_i^{(\infty)}$ defined by* (42).

(4) *If $1 < a_i + b_i < 2$, then the sequence of probabilities $p_i^{(n)}$ is swinging around $p_i^{(\infty)}$ defined by* (42).

Proof.

(1) To exhibit the threshold $p_i^{(\infty)}$ of the sequence of individual probabilities of exposure $p_i^{(n)}$, one has to keep in mind that a_i and b_i are two conditional probabilities, so, two real numbers which belong to the interval $[0, 1]$.

Then, we have:

$$0 \leq a_i + b_i \leq 2 \quad \text{and so} \quad 0 \leq 1 - a_i - b_i \leq 1\,. \tag{38}$$

As a consequence, we get the asymptotic behavior $p_i^{(\infty)}$ of the sequence $p_i^{(n)}$ given by (42) when we let N going to infinity in formula (41). This completes the proof of 1.

(2) To get the following three properties of the theorem, we modified the structure of formula (41) which gives the individual probability of exposure $p_i^{(n)}$ in a suitable form:

$$p_i^{(n)} = \frac{b_i}{a_i + b_i} + (1 - a_i - b_i)^{n-1} \left(p_i^{(1)} - \frac{b_i}{a_i + b_i} \right). \tag{39}$$

Therefore, conditions $0 < a_i + b_i < 1$ and $p_i^{(1)} > \frac{b_i}{a_i+b_i}$ lead to the decreasing property of the sequence $p_i^{(n)}$.

(3) Again, based on formula (44), the same treatment coupled to complementary conditions $0 < a_i + b_i < 1$ and $p_i^{(1)} < \frac{b_i}{a_i+b_i}$ lead to the increasing property of the sequence $p_i^{(n)}$.

(4) When one must take into account the condition $1 < a_i + b_i < 2$, the oscillatory property of the individual probabilities of exposure is due to the negative sign of the quantity $1 - a_i - b_i$.

Consequently, the sequence $p_i^{(n)}$ swings around and converges to the limit $p_i^{(\infty)}$ given by (42).

This completes the proof of the theorem. □

Remarks.

- The exposure which corresponds to conditions 3 are called " learning behavior" because it describes individuals which are in a learning process regarding a media support.
- In the contrary, the case 2 models a process called "tiring behavior" which corresponds to a "too much exposure" to a media support; these individuals are in a saturation period.
- The last case describes by condition 4 is a "swinging behavior" where the individual is neither convinced and sufficiently attracted by the media support nor saturated by its. As a consequence, the level of his exposure probabilities corresponds to an alternate dependency in relation with the media support.
- In any cases, all of these three above behaviors cannot be applied to describe media exposure in a long term period, but only for a short period. Therefore, long term description will consist to

juxtapose the above elementary short term model described by the full unsteady Markov binomial law.

Moreover, one other main difference which results from the *unsteady* sequence $(p_i^n)_{n=1,N}$ is that we cannot consider anymore relation (22) which was a consequence of the *steady* Markov process.

To obtain one more time the cover R_N for N TV spots, and therefore the asymptotic one, we remark due to the homogeneous hypothesis, we exactly conserve the same structure of formula (23) concerning the calculation of the probability for a sequence of exposure/non exposure to N successive TV spots.

More precisely, we have to write:

$$\mathrm{Prob}\{X_i^{(1)} = x_1, \ldots, X_i^{(N)} = x_N\} = p_i^{(1)} a_i^{\alpha} b_i^{\beta} (1-b_i)^{\gamma} (1-a_i)^{\delta}, \quad (40)$$

where α, β, γ and δ are defined by (24)-(27).

Remark. Formula (45) slightly differs from (23). Indeed, we took into account that the sequence of individual probabilities of exposure is not anymore constant and begins from a start potential level of individual exposure which is equal to $p_i^{(1)}$.

From the same reasons, formulas (33) and (34) describing the exposure to all or to non of the N TV spots are modified as follows:

$$\mathrm{Prob}\{X_i^{(1)} = 0, \ldots, X_i^{(N)} = 0\} = q_i^{(1)} (1-b_i)^{N-1}, \quad (41)$$

$$\mathrm{Prob}\{X_i^{(1)} = 1, \ldots, X_i^{(N)} = 1\} = p_i^{(1)} (1-a_i)^{N-1}. \quad (42)$$

Therefore, we are in position to estimate the asymptotic cover R_∞ produced by a campaign composed by N TV spots.

Theorem 3. *The asymptotic cover R_∞ of the full unsteady Markov binomial law is given by:*

$$R_\infty = 1 - \sum_{b_i=0, q_i^{(1)}=1} \pi_i - \sum_{b_{(i)}=0, q_i^{(1)} \neq 1} \pi_i q_i^{(1)}. \quad (43)$$

Proof. The demonstration looks likes to that which we proved for the full steady Markov binomial law when one remark that

$$R_N = 1 - \sum_{i=1}^{I} \pi_i \, \mathrm{Prob}\{X_i^{(1)} = 0, \ldots, X_i^{(N)} = 0\} \quad (44)$$

$$= 1 - \sum_{i=1}^{I} \pi_i q_i^{(1)} (1 - b_i)^{N-1} \tag{45}$$

$$= 1 - \sum_{b_i=0, q_i^{(1)}=1} \pi_i - \sum_{b_i=0, q_i^{(1)}\neq 1} \pi_i q_i^{(1)} - \sum_{b_i \neq 0} \pi_i q_i^{(1)} (1 - b_i)^{N-1}, \tag{46}$$

where we split again R_N into the different groups of individuals to highlight those which play a characteristic role in the estimation of the asymptotic cover R_∞.

When we let N to go to infinity, the last group of individuals in (51) vanishes and we get formula (48) to complete the proof. □

Remark. One must pay attention to observe that we can qualify in a different way the last group of individuals in (48) to distinguish the difference to that which we got in the formulation of the full steady Markov binomial model (38).

Because we lost condition (22), the characterization of people who satisfy $b_{(i)} = 0$, $q_i^{(1)} \neq 1$ only implies that $0 \leq a_i \leq 1$.

So, the asymptotic cover is finally given by:

$$R_\infty = 1 - \sum_{b_i=0, q_i^{(1)}=1} \pi_i - \sum_{b_i=0, 0\leq a_i \leq 1} \pi_i q_i^{(1)} \,. \tag{47}$$

4. Conclusions

In this paper, we presented several mathematical models describing individual media exposure regarding advertising campaigns.

We showed that when one does not consider anymore the classical binomial law (3), either by considering dependency between the different exposures on a given set of TV spots with steady Markov process (23), or by including a free individual behavior to consume a given medium by the unsteady Markov chains (45), the cover which results is significantly modified.

As a consequence, new subgroups of the population must be taken into account and one can appreciate the different level of estimation of the different models we talked.

Indeed, as a summary, we got the following asymptotic cover:

The Beta binomial: $R_\infty = 1$,

The Full binomial: $R_\infty = 1 - \sum_{p_i=0} \pi_i$, *The Full steady Markov binomial*: $R_\infty = 1 - \sum_{b_i=0, q_i=1} \pi_i - \sum_{a_i=b_i=0} \pi_i q_i$,

The Full unsteady Markov binomial: $R_\infty = 1 - \sum_{b_i=0, q_i^{(1)}=1} \pi_i - \sum_{b_i=0, 0\le a_i \le 1} \pi_i q_i^{(1)}$.

So, at our stage of investigations, between the above models, if one wants to finely describe the cover of a given advertising campaign, the recommendation is to implement the full unsteady Markov binomial model (45).

Regarding the technical problems and the associated cost to particularly estimate the two conditional probabilities (17)-(18), actually, the operational choice still is dedicated to the full binomial (1) and the beta binomial models (9) with (12).

References

[1] J. M. Agostini, Un modéle universel d'évaluation de l'audience des campagnes d'affichage, Exemples d'application á des campagnes sur emplacements fixes et emplacements mobiles, *Irep's Congres at Paris*, 1979.

[2] H. Brezis, *Analyse fonctionnelle – Théorie et applications*, Masson, 1992.

[3] J.L. Chandon, *Une étude comparee des modèles d'exposition aux supports publicitaires*, Ph.D., University of Nice, 1976.

[4] J. Chaskalovic, Pour une renaissance du média planning, *Irep's Congres at Paris*, 1993.

[5] J. Chaskalovic, An new mathematical model of exposure: the full hypernomial, *GRP's Symposium at Bruxelles*, 1994.

[6] J. Chaskalovic, Réflexions sur la description des comportements réels d'exposition média, *Irep's Congres at Paris*, 1995.

[7] P. J. Danaher, A Markov mixture model for magazine exposure, *Journal of the American Statistical Association*, (1989).

[8] P. J. Danaher, A Markov-chain model for multivariate magazine exposure distributions, *Journal of Business and Economic Statistics*, (1992).

[9] E. Deeba and S. Xie, Numerical approximation for integral equations, *International Journal of Mathematics and Mathematical Sciences*, (2004).

[10] W. Feller, *An Introduction to Probability Theory and its Applications*, Ed. Wiley, 1968.

[11] P. Hofmans, Measuring the cumulative net coverage of any combination of media, *Journal of Marketing Research*, Vol. 3 (3) (1966), pp. 269–278.

[12] J. Klerk, Solving integral equations via numerical approximations, *4th International Conference on Dynamic Systems and Applications*, 2003.

[13] R. C. MacCamy and Philip Weiss, *Numerical Approximations for Volterra Integro Equations*, Springer Verlag, Vol. 737, 1979.

[14] A. A. Markov, Extension of the limit theorems of probability theory to a sum of variables connected in a chain, reprinted in Appendix B of: R. Howard, *Dynamic Probabilistic Systems*, Volume 1: Markov Chains, John Wiley and Sons, 1971.

[15] A. Morgensztern, Frequentation de la presse - Généralisation - Effet publicitaire, *Irep's Congres at Paris*, 1970.

[16] G. Santini, *Mathematical Models and Methods for Media Research*, G.S. IT Services, 2003.

[17] F. G. Tricomi, *Integral Equations*, Interscience Publishers, 1955.

Received August, 2008