Gravitation theory for mathematical modelling in geomarketing

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Abstract

One of the objectives of geomarketing is to determine the optimal sitting of points of sale for a brand in a given territory. The aim of the optimisation is to develop sales, the flow of potential clients to the site, and to improve all the marketing indicators, which are used to describe the success of a brand within its market.

Whatever the marketing objective adopted, there are many variables within the geo-local success criteria mentioned above.

These can be socio-demographic or socio-professional variables, characteristic of structural changes in the population from one catchment area to another.

However, when considering this we must include the impact of other players in a given market, whose competitive influence is crucial to understanding each one's activity.

At the end of the 19th century, several specialists in population migration developed mathematical models in order to quantify the influence of attractive geographic points – urban centres in geography and points of sale in geomarketing.

The models were drawn up on the basis of an analogy made between the Newtonian universal force of attraction between two bodies and that which comes about between individuals of a given territory in which one or several centres of interest exist.

The object of this article is to propose new modelling perspectives based on the revolutionary theory of gravitation developed by Albert Einstein.

We will show how the general theory of relativity describes the deformation of space resulting from the presence of matter and leads to a correction of Newton's law of universal attraction, which we will use as a base to modify the laws defining the competitive influence of a point of sale in geomarketing.

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1. Introduction

Identifying distinguishing characteristics, needed to explain geo-local variations of brand penetration, is an important part of marketing thinking.

Amongst the factors potentially correlated to rates of penetration, there is a first family of variables that is exogenous to the brand, i.e., the socio-demographic and socio-professional variables characteristic of the individuals living in the given territory.

Likewise, there is a second group of variables which are endogenous to the brand, and are characteristic of the siting of its sales points, contributing to geo-local variations in the penetration rates of the advertiser. These could include the surface area of the points of sale, or their turnover.

However, the drawing up of a true marketing strategy must take on board the influence of the competition.

This dimension is useful in geomarketing as the capacity to model the influence of competition at territorial level must take into account the characteristics which are unique to this area of marketing.

In this context, it is useful to measure quantitatively the effect of the siting of the competition's points of sale, responsible for attracting a proportion of the local population.

In this way, geomarketing specialists have long identified several variables, which contribute to the modelling of this mass attraction.

If we wish to model population movements caused by the presence of a point of sale in an iris zone (a geographic unit with an average population of 2000), the main variables we will select are the geographical variable, the distance (or the travel time) between the said point of sale and the geometric centre of the iris zone, and a marketing characteristic of the point of sale such as its surface area or its turnover.

Several mathematical models have been developed to measure the attraction caused by points of sale, i.e., the movement of people, using two groups of explanatory factors and geographic, marketing.

2. Newtonian models

From the end of the 19th century, several models were developed to give a quantitative measure of the attraction on a population, spread across a territory, of a centre of attraction, i.e., a point of sale in this paper.

Reilly [1] was the first economist to propose a mathematical model in this field, basing his thinking on an analogy with Newton's concept of the universal force of attraction.

This force, which is responsible for the movement of bodies, results from the mutual attraction between matter and is proportional to the mass of the bodies and inversely proportional to the square of the distance separating the two bodies.

Likewise, and by analogy, Reilly [1] proposed that the attraction of a population caused by a point of sale followed the same type of mathematical law.

$$F_{ij} = \frac{GS_j}{r_{ij}^2}\ , \tag{1}$$

where we define the terms as follows:

F_{ij}: Influence of the Point of Sale $\boldsymbol{j}$ on the population of the Iris $\boldsymbol{i}$,
S_j: Surface area of the Point of Sale $\boldsymbol{j}$,
r_{ij}: Distance between the Point of Sale $\boldsymbol{j}$ and the Iris $\boldsymbol{i}$.

The analogy enables us to appreciate the type of trajectories between a given point in the territory and a selected point of sale.

Reilly's analogy only deals with the intensity of the universal force of attraction, so implicitly, this modelling only deals with rectilinear trajectories directed towards the point of sale.

However, two points of a phenomenological nature conflict with this description of movements on a given territory.

The first conflict is totally inherent in the reality and complexity of the layout of the national road network, and which, as a result, is in total contradiction with the simplified and rectilinear form of the trajectories underlying Reilly's model and other models resulting from it.

The second conflict with the model is more to do with the nature of the analogy made with universal attraction, and forms the starting point for our analysis.

Since this analogy is the key to understanding competitive influence, we propose in this article to describe, expand on, and refine the analogy, taking into account the significant progress made in mechanics, particularly during the 20th century.

Finally, we would point out that other models based on the Newtonian analogy have been advanced since Reilly's model.

These models describe other types of relationships between the competitive influence force and distance, by using the distance raised

to different powers or other mathematical functions explaining spatial dependence.

Denise Pumain's work [2] in particular, which presents a review of radial functions, can be consulted.

3. Short summary of theories of universal attraction

The introduction of Albert Einstein's theory of general relativity and the gravitational theory means that we can now use a better model of the theory of attraction to enable us to describe, in a subtler and more realistic way, the physics of the influence of matter.

This approach is based on the fact that the presence of a body is no longer seen as attracting another body by means of a force but rather by a deformation of the geometry of the space surrounding the body (cf. Figure 1).

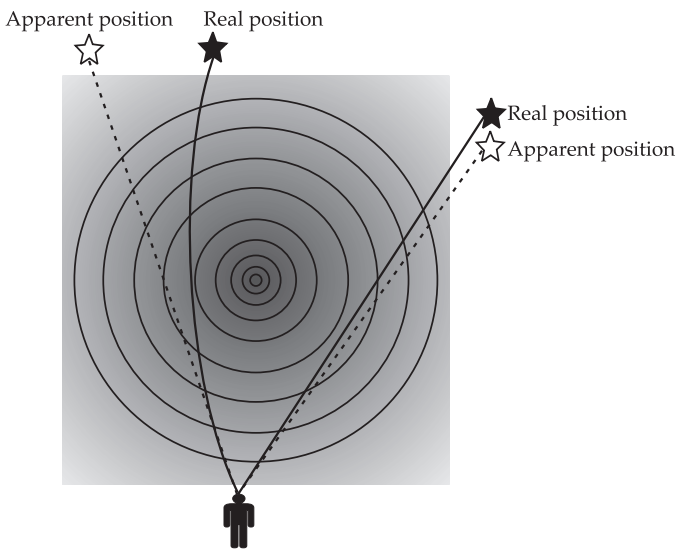

Figure 1

Trajectory deformed by the presence of a central attractive mass

The consequence of this new description of universal attraction is two-fold.

Firstly it is the deformation of the geometry of space which causes the bodies to 'slide' into the new configuration of the deformed space, so the notion of force is no longer central to this new approach to gravitation.

Secondly, the deformation of space implies that the trajectories of bodies can no longer be seen as rectilinear but *curvilinear*.

When Albert Einstein first applied this theory to the attraction of a planet by a star, his calculations showed a relatively small but consistent correction to the planet's trajectory. The reasons for the small size of the correction will be dealt with later in this article.

4. Analogy and complementary modelling in geomarketing

On the subject of the attractive influence of a point of sale in a particular country, we propose to extend the analogy taken from Einstein's work on universal attraction to the attractive potential of points of sale in a given territory.

There are many operational consequences. The main examples will be the possibility of describing the trajectories of individuals as *curvilinear* rather than rectilinear, and the possibility of creating a more precise model of competitive influence of which one application will be forecasting the geo-local variation of brand penetration rates as a function of the weight of the competition.

In order to do this, we put forward the following first principle:

Principle 1. the presence of a point of sale in a given place "deforms" an individual's trajectory, not for physical reasons, as is the case in mechanics, but because of a group of factors and items specific to geomarketing.

Whether we are considering the awareness of a brand in the territory, the power of one of its points of sale (related to surface area, turnover etc) or the intensity of the competitive environment in the territory, the presence of a point of sale must affect the trajectories of individuals who move about in its zone of influence. The size of the zone will vary according to the issues and the markets.

This is why the first principle is based on the point of view of phenomenology in geomarketing on the one hand, and an analogy with the theory of universal attraction as amended by Albert Einstein on the other hand.

5. Quantitive elements of modelling

In order to modify Reilly's law by a "relativist" correction, firstly, we need to establish the correction of Newton's law of universal attraction, taking into consideration the deformed nature of the space surrounding the attractive mass – the point of sale in geomarketing. From this point of view, we need to recall two fundamental notions of the general theory of relativity developed by Albert Einstein.

The first notion concerns the relative measurement of the distance $\boldsymbol{ds}$ in a 4-dimensional space, where the 4th dimension relates to the speed of light $\boldsymbol{c}$.

In this context, it can be shown that it is no longer possible for two observers who wish to measure the distance $\boldsymbol{ds}$, to use an expression relating to Pythagoras' theorem in a 4-dimensional space, i.e., the sum of the squares with constant coefficients.

Indeed, if one of the two observers were to move in any way, the measurement of the distance $\boldsymbol{ds}$ would be dependent on each point of the trajectory of the observer in question.

In other words, the coefficients applying to each square appearing in the equivalent formula from Pythagoras would no longer be constant but a function of the point of the observer's trajectory given by:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{2}$$

In formula (1) one must take the sum on the two indexes α and β according to Einstein's convention.

The tensor coefficients $g_{\alpha\beta}$ show, point by point, within the trajectory of the observer in question, the deformation of the geometry inherent in any movement of the observer. It is the reason why tensor g is called the metric tensor.

The second essential point in the general theory of relativity relates to the principle of equivalence, which can be expressed as follows:

The physical properties measured during any kind of movement are identical to those that can be observed on a rectilinear trajectory covered at constant speed, if the latter movement took place in the presence of a gravitational field, i.e., under the influence of matter.

In other words, this principle highlights the lack of differentiation between any kind of accelerated movement and a uniform rectilinear movement taking place in the presence of matter creating a gravitational field.

If we use the two notions we have just considered, it follows that in the presence of matter, geometry is deformed, point by point, in a manner equivalent to that which would result from any kind of movement.

From this point of view, a first modelling option, proposed by Schwarzschild [3], consists in interpreting the influence of a central attractive mass $\boldsymbol{M}$ by a deformation of the space which is identical in all directions.

Let us note that this choice can be modified at a later stage so as to introduce different trajectory curves, according to the direction selected from the central mass. This would describe, in fine, even more realistic trajectories from the geomarketing point of view.

So, in the context of gravitational influence described by Schwarzschild's geometric deformation, we must take into account the following metric expressed in spherical components:

$$g_{\alpha\beta}dx^{\alpha}dx^{\alpha} = -\left(1-\frac{2GM}{c^2r}\right)c^2dt^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{3}$$

where G is the universal constant and c the speed of the light.

Then, we can show, with the help of the general relativity equations – mainly, the geodesic equations [3], that the movement of a particle of unit mass satisfy the equation of energy:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) = \frac{\varepsilon^2 - c^4}{2c^2}\,. \tag{4}$$

Where ε is the energy of the given particle of unit mass and $V_{eff}(r)$ has the following expression:

$$V_{eff}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{c^2r^3}\,. \tag{5}$$

In formula (4), l denotes the angular momentum of the particle.

Therefore, a particle of matter of mass $\boldsymbol{m}$ would be subjected to a field of radial force which could be expressed as follows:

$$F = -\frac{GmM}{r^2} + \frac{ml^2}{r^3} - \frac{3GmMl^2}{c^2r^4}\,. \tag{6}$$

A number of points should be made if we are to analyse the values of the two corrections (in blue and red) present in the formula (5).

The first point relates to the first term which appears in formula (5) and which corresponds to Newton's universal attraction force, in accordance with the classical theory of universal gravitation.

Then, the second term in the formula (5) corresponds to that of the inherent centrifugal force in the rotation of the mass $\boldsymbol{m}$ around $\boldsymbol{M}$, which explains the change of sign compared to the first term. So, we understand this term is not relativist.

The last point concerns the third term which is purely relativist and which relates to the distance r as r^{-4}.

In other words, this correcting term is all the more intense as one approaches the attractive mass M. But this same correcting term is weighted by c^{-2}, where c corresponds to the speed of light.

The result of this is that the relativist correction is very small – insofar as the speed of light is around 300 000 km/sec – except in the immediate vicinity of the mass M, so that the dependence on r is significant enough and compensates sufficiently that of c.

Now turning to the application in geomarketing which we want to develop from the formula (5), we have to evaluate the correction that needs to be made to modify Reilly's model, based on Newton's law of universal attraction.

From this point of view, we don't keep the general framework of a particle of matter m (the individual in geomarketing) having a movement of rotation around an attractive mass M (the point of sale).

Only the case of an individual moving in the radial direction of the attractive mass M should be considered.

In other words, the force given by (5) has to be reformulated, in the case of an individual m without rotation, which results in the cancelation of parameter l.

In this case, the formula (5) is expressed as follows:

$$F = -\frac{GmM}{r^2} . \tag{7}$$

This corresponds only to Newton's force of universal attraction.

In other words, no relativist correction is made in the case of a particle m plunging without rotation towards the attractive mass M, if the influence of the mass M on the geometry of space is modeled by Schwarzschild's deformation.

So, we were lead to consider an alternative to highlight a relativist correction which would remain in the case of movement of a mass m without rotation around an attractive mass M.

The challenge is to find an alternative modelling to that of Schwarzchild's in order to take into account this relativist correction.

Bearing this in mind, we identified the work of the mathematician Roy Kerr, who, in 1963, described the deformation of the geometry of

space in the vicinity of a rotating black hole.

Kerr's metric is given by:

$$g_{\alpha\beta}dx^{\alpha}dx^{\beta} = -\left(1-\frac{2GMr}{c^2\rho^2}\right)c^2dt^2 - \frac{4GMar\sin^2\theta}{c^2\rho^2}cdtd\varphi + \frac{\rho^2}{\Delta}dr^2 + \rho^2d\theta^2 + \left(r^2+a^2+\frac{2GMa^2r\sin^2\theta}{c^2\rho^2}\right)\sin^2\theta d\varphi^2 . \quad (8)$$

Where:

$$\Delta = r^2 - \frac{2GM}{c^2}r + a^2 \quad (9)$$

and

$$\rho^2 = r^2 + a^2\cos^2\theta . \quad (10)$$

Parameter a describes in formula (7) the kinetic moment of the rotating black hole.

Using this work, we can show, another time, with the help of the general relativity equations, that the force of attraction of a central mass $\boldsymbol{M}$ can be written in the form:

$$F = -\frac{GmM}{r^2} + \frac{M[l^2 - a^2(\epsilon^2/c^2 - c^2)]}{r^3} - \frac{3GmM}{c^2r^4}\left(l - \frac{a\epsilon}{c}\right)^2 , \quad (11)$$

where we can note that two supplementary parameters appear compared to our previous model which was based on Schwarzschild's work.

These are: parameter ε, characterising the energy of the mass $\boldsymbol{m}$, and parameter $\boldsymbol{a}$, associated with the rotation of the central black hole.

If we want to describe a radial trajectory of the mass $\boldsymbol{m}$ towards the black hole $\boldsymbol{M}$, we have to make $l = 0$ in formula (10) and then obtain:

$$F = -\frac{GmM}{r^2} - \frac{ma^2(\epsilon^2/c^2 - c^2)}{r^3} - \frac{3GmMa^2\epsilon^2}{c^4r^4} . \quad (12)$$

Therefore, two purely relativist corrections appear whose relationship regarding the radial distance are $\boldsymbol{r}^{-3}$ and $\boldsymbol{r}^{-4}$, respectively.

This result confirms the necessity to take the two relativist terms into consideration, especially as the description of the motion relating to the trajectory of the mass $\boldsymbol{m}$ to the vicinity of the black hole $\boldsymbol{M}$.

We can now propose, by analogy, a modified Reilly's law as follows:

$$F_{ij} = \frac{GS_j}{r_{ij}^2} + \frac{a^2(\epsilon^2/c^2 - c^2)}{r_{ij}^3} + \frac{3GS_ja^2\epsilon^2}{c^4r_{ij}^4} . \quad (13)$$

We note that the new model governed by formula (12) requires the determination of the 4 parameters $(\boldsymbol{G}, \boldsymbol{a}, \boldsymbol{\varepsilon}, \boldsymbol{c})$.

These parameters must be determined using appropriate statistic non-linear adjustment techniques.

However, the analogy that we propose between the results of general relativity and geomarketing may raise a problem of interpretation, which we now wish to address.

If the parameters $(\boldsymbol{G}, \boldsymbol{a}, \boldsymbol{\varepsilon})$ can find equivalent interpretations in geomarketing to that of their equivalents in mechanics, we could ask what meaning could be given in geomarketing to parameter c describing the speed of light in general relativity?

Concerning the interpretation of the constant $\boldsymbol{G}$, Claude Grasland, Professor of Geography at the University of Paris 7, UFR Geography History Social Science, explains: "...this parameter ensures the conservation of the general total of flows by stating that the sum of the flows observed must be equal to the sum of the estimated flows...".

Concerning parameter $\boldsymbol{a}$, associated with the rotation of the black hole in astrophysics, it is a question of completing the description of the point of sale, which has so far been described in terms of its surface area S_j.

This may involve the turnover or any other marketing indicator which provides a relevant analogy to the "rotation" of the point of sale concerned.

The parameter $\boldsymbol{\varepsilon}$, describing the energy of the mass $\boldsymbol{m}$ orbiting in the field of the attractive mass $\boldsymbol{M}$, could describe the characteristics of the individuals residing in the Iris $\boldsymbol{i}$, and who are potential customers of the point of sale $\boldsymbol{j}$.

This could then correspond to the socio-demographic or socio-professional characteristics of the individuals residing in the area of influence of the point of sale.

Now, concerning parameter $\boldsymbol{c}$, we find ourselves facing a real and fundamental problem concerning the capacity to transpose such a concept – the speed of light – for an application modelling competitive influence in geomarketing.

Does there really exist an equivalent to the speed of light $\boldsymbol{c}$ in geomarketing? In response to this question, we suggest two lines of enquiry, the main points of which follow.

In physics, the speed of light constitutes an insurmountable barrier to the movement of any material body. More specifically, one of the principles of restricted relativity can be stated as follows:

Principle of restricted relativity. Material bodies cannot move faster than the speed of light c.

Consequently, and in order to transpose this principle into the geomarketing framework, we propose the following principle:

Principle 2. All movement on the national territory is governed by the Rules of the Road and cannot exceed the French speed limit of 130 Km/h.

However, an alternative to this way of thinking may prove necessary as the analogy with physics has limitations. This is underlined by Denise Pumain in her book "Les interactions spatiales" (spatial interactions) [2]: ... "The intervention necessary in the model's formula emphasises that it is not an application of the Newtonian model in the social field: moreover, it is hard to see what would justify it in theory!..."

It is for this reason that an alternative method, allowing us to estimate parameter c by statistical adjustment as a function of real data, could constitute a complementary solution to that which we have proposed, based on the analogy with mechanics.

In any event only numerical experimentation will decide between these two exploratory paths, leading to the numerical estimation of the parameters of the model.

In order to do this, we must evaluate the importance of the "relativist" contribution according to each of the methods from the point of view of a more exact and realistic description of competitive influence.

Finally, let us note that in the new law of competitive influence, the "relativist" correction is greater as we get closer to the point of sale, as it is inversely proportional to the power of three and four of that distance.

This would seem to indicate that the Reilly model does not apply to the vicinity of points of sale when describing their influence.

6. Outlook

There is a lot at stake in the modelling of the interactive power of points of sale in geomarketing. Any advertiser whose marketing strategy is dependent on their territorial coverage must define their distribution network in relation to the geomarketing characteristics of the competition.

In fact, the capacity to evaluate, as precisely as possible, the influence of the competitive presence is a determining factor in geo-local variations

of the penetration rates of a brand, its turnover, or even the flow of customers in a shop. The modelling we have proposed contributes to the debate and aims to provide as precise a description as possible of competitive influence in geomarketing.

The work underway aims to estimate the parameters of the "relativist" gravitation model with the help of non-linear statistical adjustment techniques.

In any event, it is reasonable to expect that the new modelling will produce numerical results which are all the more reliable as we are trying to estimate the competitive influence in the close vicinity of a given point of sale.

In this respect, we can note that the theory of gravitational fields in general relativity predicts that there is a critical range, called the "gravitational range", inside of which nothing, including light, can escape the influence of the attracting mass.

If a planet or any other object were to enter this critical attraction zone, it could not escape, becoming a prisoner of the gravitational field of the central mass.

Consequently, it is reasonable to ask the following question: What about this new way of modelling in geomarketing? Is there a critical distance inside of which individuals would be "trapped" because of the intensity of the "mass" of the point of sale?

These questions will be answered when the parameters of the model have been evaluated as we indicated at the beginning of this paragraph.

If we were to take into account the influence of a point of sale using models based on the behaviour of masses in a gravitational field we would expect to get similar results to those described. The intensity of this influence could reach a critical level within a zone which would be useful to evaluate.

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