An optimal first-order Taylor-like formula with a minimized remainder

Joël Chaskalovic, Franck Assous

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Abstract

In this paper, we derive an optimal first-order Taylor-like formula. In a seminal paper [14], we introduced a new first-order Taylor-like formula that yields a reduced remainder compared to the classical Taylor's formula. Here, we relax the assumption of equally spaced points in our formula. Instead, we consider a sequence of unknown points and a sequence of unknown weights. Then, we solve an optimization problem to determine the best distribution of points and weights that ensures that the remainder is as minimal as possible.

keywords: Taylor's theorem, Taylor-like formula, Error estimate, Interpolation error, Approximation error, Finite elements.

1 Introduction

Even today, improving the accuracy of approximations remains a challenging problem in numerical analysis. In this context, Taylor's formula plays a crucial role in various domains, especially when one considers error estimates in numerical analysis to assess the accuracy of a numerical approximation method (for example, see [21] for finite element methods).

This challenge becomes even more crucial when comparing the relative accuracy between two given numerical methods. All error estimates share a common structure, whether applied to the finite elements method [5], [17], numerical approximations of ordinary differential equations [15], or to quadrature formulas used for approximating integrals [8].

Let us specify these ideas in this context of numerical integration. Consider, for instance, a composite quadrature rule of order k. For a given interval $[a, b]$, let f be a function in $C^{k+1}([a, b])$. The corresponding error of the composite quadrature rule can be expressed as (refer to, e.g., [3], [6] or [15]), for a non-zero integer N:

$$\left| \int_a^b f(x)dx - \sum_{i=0}^{N} \lambda_i f(x_i) \right| \leq C_k \, h^{k+1} .$$

In this formula, h denotes the size of the $N+1$ equally spaced panels $[x_i, x_{i+1}]$, $0 \leq i \leq N$, that discretize the interval $[a, b]$, and λ_i are $N+1$ real numbers. Moreover, C_k is an unknown constant, independent of h, but dependent on f and k. This constant is directly linked to the uncertainty associated with the remainder of Taylor's formula.

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Usually, to overcome the lack of information associated with the unknown value of the left-hand side which belongs to the interval $[0, C_k h^{k+1}]$, only the asymptotic convergence rate comparison is considered. This comparison allows us to assess the relative accuracy between two numerical quadratures of order k_1 and $k_2, (k_1 < k_2)$, when h tends to zero.

However, comparing the asymptotic convergence rate is no longer useful when evaluating two composite quadratures rules for a fixed value of h, as is common in various applications. Therefore, we focus our attention to minimize the constants C_k by improving the estimation of the remainder involved in Taylor's formula.

In this context, several approaches have been proposed to determine a way to enhance the accuracy of approximation. For example, within the framework of numerical integration, we refer the reader to [4], [7] or [16], and references therein. From another point of view, due to the lack of information, heuristic methods were considered, basically based on a probabilistic approach, see for instance [1], [2], [18], [19] or [8], [9] and [10]. This allows to compare different numerical methods, and more precisely finite element, for a given fixed mesh size, [11].

In this context, we recently developed a first-order Taylor-like formula in [14] and a second-order Taylor-like formula in [13]. The goal was to minimize the corresponding remainder by transferring part of the numerical weight of this remainder to the polynomial involved in the Taylor expansion.

In both of these cases, we *a priori* introduced a linear combination of f' (and f'' in [13]) computed at equally spaced points in $[a, b]$, and we determined the corresponding weights in order to minimize the remainder. We demonstrated that the associated upper bound in the error estimate is $2n$ times smaller than the classical one for the first-order Taylor's theorem, and $3/16n^2$ times smaller than the classical second-order Taylors's formula.

In this paper, we relax the assumption of equally spaced points and consider a sequence of unknown points in the interval $[a, b]$, where a given function f needs to be evaluated. Simultaneously, we introduce a sequence of unknown weights to be determined with the goal of minimizing the remainder. Then, we will prove that the remainder of the corresponding first-order expansion is minimized when the points between a and b are equally spaced.

The paper is organized as follows. In Section 2, we present a new first-order Taylor-like formula built on a sequence of given points $x_k, (k = 1, \dots, n-1)$, in $[a, b]$, and given weights $\omega_k, (k = 0, \dots, n)$. In Section 3 we derive the main results of this paper which deal with the optimal choice of points x_k and weights ω_k that enable us to minimize the corresponding remainder of the first-order Taylor-like formula. Concluding remarks follow.

2 A first-order Taylor-like theorem

Let us begin by recalling the well-known first order Taylor formula [22]. For two reals a and b, $a < b$, we consider a function $f \in \mathcal{C}^2([a, b])$. Then, there exists $(m_2, M_2) \in \mathbb{R}^2$ such that, for all $x \in [a, b]$,

$$m_2 \leqslant f''(x) \leqslant M_2$$

and we have

$$f(b) = f(a) + (b-a)f'(a) + (b-a)\epsilon_{a,1}(b), \tag{1}$$

with

$$\lim_{b \to a} \epsilon_{a,1}(b) = 0,$$

and

$$\frac{(b-a)}{2} m_2 \leqslant \epsilon_{a,1}(b) \leqslant \frac{(b-a)}{2} M_2.$$

In a previous paper [14], we derived a first-order Taylor-like formula with the aim of minimizing the classical remainder $\epsilon_{a,1}(b)$. More precisely, we proved the following result:

Theorem 2.1 *Let f be a real mapping defined on $[a,b]$ which belongs to $\mathcal{C}^2([a,b])$, such that: $\forall x \in [a,b], -\infty < m_2 \leqslant f''(x) \leqslant M_2 < +\infty$.*

Then, for a given non-zero integer n, we have the following first-order expansion:

$$f(b) = f(a) + (b-a)\left(\frac{f'(a)+f'(b)}{2n} + \frac{1}{n}\sum_{k=1}^{n-1} f'\left(a + k\frac{(b-a)}{n}\right)\right) + (b-a)\epsilon_{a,n+1}(b), \qquad (2)$$

where

$$|\epsilon_{a,n+1}(b)| \leqslant \frac{(b-a)}{8n}(M_2 - m_2).$$

Moreover, for an a priori *choice of regularly spaced points $a + k\frac{(b-a)}{n}$, the remainder $\epsilon_{a,n+1}(b)$ is minimum.*

In the following of the paper, our goal is to relax the assumption of *a priori* equidistant points to determine the optimal set of points $(x_k)_{k=0,n} \in [a,b]$, along with the associated weights $(\omega_k)_{k=0,n}$. This determination will enable us to minimize the remainder $\epsilon_{a,n+1}(b)$ in (2).

To derive the main result below, we first introduce the function ϕ defined by

$$\begin{aligned} \phi: \quad [0,1] &\longrightarrow \mathbb{R} \\ t &\longmapsto f'(a + t(b-a)), \end{aligned} \qquad (3)$$

that satisfies $\phi(0) = f'(a)$ and $\phi(1) = f'(b)$. Moreover, we proved in [14] that the remainder $\epsilon_{a,1}(b)$ introduced in (1) satisfies the following result:

Proposition 2.2 *The remainder $\epsilon_{a,1}(b)$ in formula (1) can be expressed as*

$$\epsilon_{a,1}(b) = \int_0^1 (1-t)\phi'(t)dt\,. \qquad (4)$$

For a given integer $n \in \mathbb{N}^*$, we consider the set of points $(x_k)_{k=0,n}$ in the interval $[a,b]$, where $x_0 = a$ and $x_n = b$, on the one hand, and a set of real weights $(\omega_k)_{k=0,n}$, on the other hand.

Then, we define the quantity $\epsilon_n^*(a,b)$ by the formula

$$f(b) = f(a) + (b-a)\left(\sum_{k=0}^{n} \omega_k f'(x_k)\right) + (b-a)\epsilon_n^*(a,b), \qquad (5)$$

where the two sequences $(x_k)_{k=1,n-1}$ and $(\omega_k)_{k=0,n}$ are to be determined to minimize the remainder $\epsilon_n^*(a,b)$.

For the upcoming result, we need to introduce some notations. We denote by $t_k, (k = 0, \dots, n)$, a sequence of real numbers that allows us to represent the points x_k in $[a, b]$ as a barycentric combination of a and b, that is:

$$x_k = a + t_k(b - a),\ (0 \le t_k \le 1). \tag{6}$$

We also introduce S_k as the partial sum of the weights ω_j, $0 \le j \le k$,

$$S_k = \sum_{j=0}^{k} \omega_j. \tag{7}$$

Now, we can prove the following result:

Theorem 2.3 *Let f be a real mapping defined on $[a, b]$ which belongs to $\mathcal{C}^2([a, b])$. Then, the remainder $\epsilon_n^*(a, b)$ defined by (5) satisfies:*

$$\frac{(b-a)}{2}\sum_{k=0}^{n-1}\Big[m_2(S_k - t_k)^2 - M_2(S_k - t_{k+1})^2\Big] \le \epsilon_n^*(a,b) \le \frac{(b-a)}{2}\sum_{k=0}^{n-1}\Big[M_2(S_k - t_k)^2 - m_2(S_k - t_{k+1})^2\Big]. \tag{8}$$

Proof : Using first the function ϕ defined in (3), formulas (1) and (5) lead to

$$\frac{f(b) - f(a)}{b - a} = \phi(0) + \epsilon_{a,1}(b) = \sum_{k=0}^{n} \omega_k \phi(t_k) + \epsilon_n^*(a, b),$$

which can be re-written as:

$$\begin{aligned}
\epsilon_n^*(a,b) &= \phi(0) + \epsilon_{a,1}(b) - \sum_{k=0}^{n} \omega_k \phi(t_k), \\
&= \phi(0) + \int_0^1 (1-t)\phi'(t)dt - \sum_{k=0}^{n} \omega_k \phi(t_k), \\
&= \phi(1) - \int_0^1 t\phi'(t)dt - \sum_{k=0}^{n} \omega_k \phi(t_k), \\
&= \phi(1) - \int_0^1 t\phi'(t)dt + \sum_{k=0}^{n} \omega_k \left(\phi(1) - \phi(t_k)\right) - \sum_{k=0}^{n} \omega_k \phi(1), \\
&= \left(1 - \sum_{k=0}^{n} \omega_k\right)\phi(1) - \int_0^1 t\phi'(t)dt + \sum_{k=0}^{n} \omega_k \left(\phi(1) - \phi(t_k)\right).
\end{aligned} \tag{9}$$

Let us assume that weights $(\omega_k)_{k=0,n}$ fulfill the condition

$$\sum_{k=0}^{n} \omega_k = 1\,, \tag{10}$$

equation (9) can be expressed as

$$
\begin{aligned}
\epsilon_n^*(a,b) &= -\int_0^1 t\phi'(t)dt + \sum_{k=0}^{n} \omega_k \left(\phi(1) - \phi(t_k)\right), \\
&= -\int_0^1 t\phi'(t)dt + \sum_{k=0}^{n} \omega_k \int_{t_k}^1 \phi'(t)dt, \\
&= -\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} t\phi'(t)dt + \sum_{k=0}^{n} \omega_k \int_{t_k}^1 \phi'(t)dt. \qquad (11)
\end{aligned}
$$

Considering the second term of (11), it can be transform as follows:

$$
\begin{aligned}
\sum_{k=0}^{n} \omega_k \int_{t_k}^1 \phi'(t)dt &= \int_{t_0}^1 \omega_0\phi'(t)dt + \int_{t_1}^1 \omega_1\phi'(t)dt + \cdots + \int_{t_{n-1}}^1 \omega_{n-1}\phi'(t)dt, \\
&= \int_{t_0}^{t_1} \omega_0\phi'(t)dt + \int_{t_1}^{t_2} (\omega_0 + \omega_1)\phi'(t)dt + \cdots + \int_{t_{n-1}}^1 (\omega_0 + \cdots + \omega_{n-1})\phi'(t)dt. \\
&= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S_k\, \phi'(t)dt, \qquad (12)
\end{aligned}
$$

where S_k is defined by (7).

Then, using (12) in (11) enables to write $\epsilon_n^*(a,b)$ as

$$
\epsilon_n^*(a,b) = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (S_k - t)\phi'(t)dt. \qquad (13)
$$

In the estimations below, we will use that

$$
\forall\, t \in [0,1], \phi'(t) = (b-a)f''(a + t(b-a)), \quad \text{and } \forall\, x \in [a,b], m_2 \leqslant f''(x) \leqslant M_2.
$$

Next, to derive a double inequality on $\epsilon_n^*(a,b)$, we consider the three following cases, depending on the location of S_k related to the interval $[t_k, t_{k+1}]$.

1. If $t_k \leq S_k \leq t_{k+1}$, the integral in (13) can be decomposed as follows:

$$
\int_{t_k}^{t_{k+1}} (S_k - t)\phi'(t)dt = \int_{t_k}^{S_k} (S_k - t)\phi'(t)dt + \int_{S_k}^{t_{k+1}} (S_k - t)\phi'(t)dt. \qquad (14)
$$

Now, using that $(S_k - t)$ is positive on $[t_k, S_k]$, and negative on $[S_k, t_{k+1}]$, we can write

$$
\begin{cases}
(b-a)m_2 \displaystyle\int_{t_k}^{S_k} (S_k - t)dt \leq \int_{t_k}^{S_k} (S_k - t)\phi'(t)dt \leq (b-a)M_2 \int_{t_k}^{S_k} (S_k - t)dt, \\
(b-a)M_2 \displaystyle\int_{S_k}^{t_{k+1}} (S_k - t)dt \leq \int_{S_k}^{t_{k+1}} (S_k - t)\phi'(t)dt \leq (b-a)m_2 \int_{S_k}^{t_{k+1}} (S_k - t)dt.
\end{cases}
$$

Summing up these two relations, we obtain first that

$$\int_{t_k}^{t_{k+1}} (S_k - t)\phi'(t)dt \le (b-a)M_2 \int_{t_k}^{S_k} (S_k - t)dt + (b-a)m_2 \int_{S_k}^{t_{k+1}} (S_k - t)dt,$$

and also

$$\int_{t_k}^{t_{k+1}} (S_k - t)\phi'(t)dt \ge (b-a)m_2 \int_{t_k}^{S_k} (S_k - t)dt + (b-a)M_2 \int_{S_k}^{t_{k+1}} (S_k - t)dt\,.$$

By simply computing the integrals involved in these relations, we obtain, in that case, the estimate

$$\frac{(b-a)}{2}\Big(m_2(S_k-t_k)^2 - M_2(S_k-t_{k+1})^2\Big) \le \int_{t_k}^{t_{k+1}} (S_k-t)\phi'(t)dt \le \frac{(b-a)}{2}\Big(M_2(S_k-t_k)^2 - m_2(S_k-t_{k+1})^2\Big). \tag{15}$$

2. If $S_k \ge t_{k+1}$ then $S_k - t \ge 0$ for all $t \in [t_k, t_{k+1}]$, and we have:

$$(b-a)m_2 \int_{t_k}^{t_{k+1}} (S_k - t)dt \le \int_{t_k}^{t_{k+1}} (S_k - t)\phi'(t)dt \le (b-a)M_2 \int_{t_k}^{t_{k+1}} (S_k - t)dt\,.$$

This yields

$$\frac{(b-a)m_2}{2}\Big((S_k-t_k)^2 - (S_k-t_{k+1})^2\Big) \le \int_{t_k}^{t_{k+1}} (S_k-t)\phi'(t)dt \le \frac{(b-a)M_2}{2}\Big((S_k-t_k)^2 - (S_k-t_{k+1})^2\Big),$$

which also leads to (15), by simply using that $m_2 \le M_2$.

3. If $S_k \le t_k$ then $S_k - t \le 0$ for all $t \in [t_k, t_{k+1}]$, and in the same way as above, we get

$$\frac{(b-a)M_2}{2}\Big((S_k-t_k)^2 - (S_k-t_{k+1})^2\Big) \le \int_{t_k}^{t_{k+1}} (S_k-t)\phi'(t)dt \le \frac{(b-a)m_2}{2}\Big((S_k-t_k)^2 - (S_k-t_{k+1})^2\Big),$$

that also gives estimate (15).

Hence, in all cases, we arrive at the same estimate (15). Finally, by summing over (15) over all values of k from 0 to $n-1$ in (15), we get inequalities (8) for the remainder $\epsilon_n^*(a,b)$, which concludes the proof. ■

With the aim of minimizing $\epsilon_n^*(a,b)$, we introduce the function χ defined by

$$\chi = \frac{(b-a)(M_2-m_2)}{2} \sum_{k=0}^{n-1} \left[\left(\sum_{j=0}^{k} \omega_j - t_k \right)^2 + \left(\sum_{j=0}^{k} \omega_j - t_{k+1} \right)^2 \right], \tag{16}$$

which represents to the difference between the right-hand side and the left-hand side in (8).

As a consequence, in the next section, we will minimize function χ which depends on $2n-1$ variables, namely $(t_1, \ldots, t_{n-1}, \omega_0, \ldots, \omega_{n-1})$. This is because the boundary points are known ($x_0 = a$ corresponding to $t_0 = 0$ and $x_n = b$ corresponding to $t_n = 1$), on the one hand, and due to relation (10) which will determine ω_n, on the other hand.

3 The optimal first order Taylor-like formula

We begin this section by deriving a lemma that provides a necessary condition for the function χ to have an extremum at the point $(t_1, \dots, t_{n-1}, \omega_0, \dots, \omega_{n-1})$.

Lemma 3.1 *Let* $(t_1, \dots, t_{n-1}, \omega_0, \dots, \omega_{n-1})$ *be an extremum of function* χ.

Then, we have:

$$\begin{cases} S_k = \dfrac{1}{2}(t_k + t_{k+1}), \, (k = 0, \dots, n-1), & (17) \\ t_k = S_{k-1} + \dfrac{\omega_k}{2}, \quad (k = 1, \dots, n-1). & (18) \end{cases}$$

Proof : The necessary conditions that guarantee that $(t_1, \dots, t_{n-1}, \omega_0, \dots, \omega_{n-1})$ is an extremum is written as (see for instance [20])

$$\forall k = 0, \dots, n-1 : \frac{\partial \chi}{\partial \omega_k} = 0 \text{ and } \forall k = 1, \dots, n-1 : \frac{\partial \chi}{\partial t_k} = 0.$$

Regarding first the dependence of the function χ on the variables ω_k, $(k = 0, \dots, n-1)$, the conditions $\dfrac{\partial \chi}{\partial \omega_k} = 0$ are expressed as

$$\sum_{m=k}^{n-1} \Big[2S_m - t_m - t_{m+1} \Big] = 0 \,,$$

or equivalently

$$\sum_{m=k}^{n-1} S_m = \frac{1}{2} \sum_{m=k}^{n-1} (t_m + t_{m+1}) \,. \qquad (19)$$

Since this system of equations is triangular, it can be easily solved. Writing two consecutive equations for a given $k \in \{0, \dots, n-2\}$ leads to

$$\begin{aligned} S_k + S_{k+1} + \cdots + S_{n-1} &= \frac{1}{2}\big[(t_k + t_{k+1}) + (t_{k+1} + t_{k+2}) + \cdots + (t_{n-1} + t_n)\big], \\ S_{k+1} + \cdots + S_{n-1} &= \frac{1}{2}\big[(t_{k+1} + t_{k+2}) + \cdots + (t_{n-1} + t_n)\big] \,, \end{aligned}$$

which readily gives, by difference,

$$S_k = \frac{1}{2}(t_k + t_{k+1}) \,,$$

the case $k = n-1$ corresponding directly to the last equation of the system (19).

Now, to study the dependence of the function χ on the variables t_k, $\forall k = 1, \dots, n-1$, we expand formula (16), using that $t_0 = 0$ and $t_n = 1$. We obtain that

$$\begin{aligned}
\frac{2}{(b-a)(M_2-m_2)}\chi &= \sum_{k=0}^{n-1}\left[\Big(\omega_0+\cdots+\omega_k-t_k\Big)^2+\Big(\omega_0+\cdots+\omega_k-t_{k+1}\Big)^2\right]\\
&= (\omega_0-0)^2+(\omega_0-t_1)^2\\
&\quad+(\omega_0+\omega_1-t_1)^2+(\omega_0+\omega_1-t_2)^2\\
&\quad+\dots\\
&\quad+(\omega_0+\cdots+\omega_k-t_{k-1})^2+(\omega_0+\cdots+\omega_k-t_k)^2\\
. &\quad+\dots\\
&\quad+(\omega_0+\cdots+\omega_{n-1}-t_{n-1})^2+(\omega_0+\cdots+\omega_{n-1}-1)^2\,.
\end{aligned}$$

So, by taking the derivative of the function χ with respect to t_k, we obtain, for each $k = 1, \dots, n-1$:

$$\frac{\partial\chi}{\partial t_k}=0 \quad\Leftrightarrow\quad 2(\omega_0+\cdots+\omega_{k-1}-t_k)+2(\omega_0+\cdots+\omega_k-t_k)=0\,.$$

This can be expressed as

$$S_{k-1}+S_k-2t_k=0$$

that is, using the definition (7) of S_k

$$2t_k = 2S_{k-1}+\omega_k\,,$$

which corresponds to (18). ■

From Lemma 3.1, we can now state the main result of this paper:

Theorem 3.2 *Let f be a real function defined on $[a,b]$ that belongs to $\mathcal{C}^2([a,b])$. Then, the optimal unknown weights $(\omega_m)_{m=0,n}$ together with the optimal set of points $(x_m)_{m=1,n-1}$ determined by the sequence of real numbers $(t_m)_{m=1,n-1}$ that minimizes the remainder $\epsilon_n^*(a,b)$ defined by (5) are given by:*

$$\omega_0=\omega_n=\frac{1}{2n} \;\text{ and }\; \omega_k=\frac{1}{n}, \forall k=1,\dots,n-1, \tag{20}$$

$$t_k=\frac{k}{n} \;\text{ and }\; x_k=a+\frac{k}{n}(b-a),\ \forall k=1,\dots,n-1. \tag{21}$$

As a result, the corresponding optimal first-order Taylor-like formula is given by the following expression:

$$f(b)=f(a)+(b-a)\left(\frac{f'(a)+f'(b)}{2n}+\frac{1}{n}\sum_{k=1}^{n-1}f'\left(a+k\frac{(b-a)}{n}\right)\right)+(b-a)\epsilon_n^*(a,b), \tag{22}$$

with

$$|\epsilon_n^*(a,b)|\leqslant\frac{(b-a)}{8n}(M_2-m_2). \tag{23}$$

Proof :

— We begin by proving that ω_k is constant for all values of k belonging to $\{1, \dots, n-1\}$

From (17) and (18), we have

$$2S_k = t_k + t_{k+1} = S_{k-1} + \frac{\omega_k}{2} + S_k + \frac{\omega_{k+1}}{2},\ \forall k = 1, \dots, n-2,$$

that yields

$$S_k - S_{k-1} := \omega_k = \frac{\omega_k + \omega_{k+1}}{2}.$$

Then,

$$\omega_{k+1} = \omega_k,\ \forall k = 1, \dots, n-2, \tag{24}$$

that corresponds to

$$\omega_1 = \dots = \omega_{n-1}.$$

— We will now establish the relation between ω_0 and ω_k, for $k = 1, \dots, n-1$.

Firstly, let us write (17) for $k = 0$ and (18) for $k = 1$. Using that $t_0 = 0$, we obtain that

$$\begin{aligned} 2S_0 &= t_0 + t_1 = t_1, \\ t_1 &= S_0 + \frac{\omega_1}{2} = \omega_0 + \frac{\omega_1}{2}, \end{aligned}$$

from which we get

$$\omega_1 = 2\omega_0\,.$$

This allows us to conclude, using (24), that

$$\omega_1 = \omega_2 = \dots = \omega_{n-1} = 2\omega_0. \tag{25}$$

— Let us compute now the value of ω_0.

To this end, we write (17) and (18) for $k = n-1$. Given that $t_n = 1$ and using (25), this yields

$$\begin{aligned} 2S_{n-1} &= t_{n-1} + t_n = t_{n-1} + 1, \\ &= S_{n-2} + \frac{\omega_{n-1}}{2} + 1. \end{aligned}$$

Then, substituting the expressions of S_{n-1} and S_{n-2}, we get with (25) that

$$2\omega_0 + 4(n-1)\omega_0 = \omega_0 + 2(n-2)\omega_0 + \frac{\omega_{n-1}}{2} + 1,$$

that leads to

$$2(2n+1)\omega_0 - 2 = \omega_{n-1} = 2\omega_0$$

that is

$$\omega_0 = \frac{1}{2n}. \tag{26}$$

— It remains now to compute the value of ω_n. Using (25) and (26) gives directly that

$$\omega_1 = \omega_2 = \dots = \omega_{n-1} = \frac{1}{n}. \tag{27}$$

Finally, the value of ω_n is obtained from relation (10), that is

$$\omega_n = 1 - \omega_0 - \sum_{k=1}^{n-1} \omega_k = 1 - \frac{1}{2n} - \frac{n-1}{n} = \frac{1}{2n}.$$

— Let us now consider the values of the t_k, for $k = 1\ldots, n-1$.

Using the expressions (18) of the optimal t_k, together with the expressions (26) and (27) of the ω_k, we obtain that

$$\begin{aligned} t_k &= \omega_0 + \sum_{j=1}^{k-1} \omega_j + \frac{\omega_k}{2}, (k = 1, \ldots, n-1) \\ &= \frac{1}{2n} + \frac{k-1}{n} + \frac{1}{2n} = \frac{k}{n}, \end{aligned}$$

that yields with (6), the following expressions of the optimal points x_k:

$$x_k = a + \frac{k}{n}(b-a), (k = 1, \ldots, n-1). \tag{28}$$

— We conclude the proof of this theorem by determining the optimal lower and upper bounds of the remainder $\epsilon_n^*(a,b)$ given in (8).

1. We first evaluate the quantity $\displaystyle\sum_{k=0}^{n-1}(S_k - t_k)^2$:

   Using the expression (20) of the ω_j, we obtain that

   $$S_k = \sum_{j=0}^{k} \omega_j = \omega_0 + \sum_{j=1}^{k} \omega_j = \frac{1}{2n} + \frac{k}{n} = \frac{2k+1}{2n}, (k = 0, \ldots, n-1)\,,$$

   so that, together with the expression (21) of the t_k, we have

   $$\sum_{k=0}^{n-1}(S_k - t_k)^2 = \sum_{k=0}^{n-1}\left(\frac{2k+1}{2n} - \frac{k}{n}\right)^2 = \frac{1}{4n}. \tag{29}$$

2. Similarly, we evaluate the quantity $\displaystyle\sum_{k=0}^{n-1}(S_k - t_{k+1})^2$. The same arguments yield

   $$\begin{aligned} \sum_{k=0}^{n-1}(S_k - t_{k+1})^2 &= \sum_{k=0}^{n-2}(S_k - t_{k+1})^2 + (S_{n-1} - t_n)^2 \\ &= \sum_{k=0}^{n-2}\left(\frac{2k+1}{2n} - \frac{k+1}{n}\right)^2 + \left(\frac{2n-1}{2n} - 1\right)^2 \\ &= \sum_{k=0}^{n-2}\left(\frac{-1}{2n}\right)^2 + \frac{1}{4n^2} = \frac{1}{4n}. \end{aligned} \tag{30}$$

Therefore, combining (29) and (30), we obtain optimal lower bound and upper bounds in (8) for the remainder $\epsilon_n^*(a,b)$ as follows:

$$\frac{(b-a)}{8n}(m_2 - M_2) \leq \epsilon_n^*(a,b) \leq \frac{(b-a)}{8n}(M_2 - m_2),$$

which exactly corresponds to the result derived in [14].

■

4 Conclusion

In this paper, we have derived a new first-order Taylor-like formula constructed as a linear combination of the first derivatives of a given function, evaluated at specified points $x_k, (k = 1, \dots, n-1)$, within an interval $[a, b]$. These points are weighted by real numbers $\omega_k, (k = 0, \dots, n)$. Unlike the approach in [14], the positions of these points and their corresponding weights were not fixed *a priori*. In particular, we relax the assumption of equally spaced points.

Our aim was to determine the optimal positions of these points together with the weights for obtaining the "best formula", in the sense that the corresponding remainder is as small as possible. To achieve this, we have established an initial result that provides upper and lower bounds for the remainder.

Then, we proved the existence of an optimal set of points and weights that minimize the remainder in the first-order Taylor-like formula. This result corresponds to the formula we derived in [14], where we explicitely set *a priori* the values of the points $x_k, (k = 1, \dots, n-1)$, as uniformly distributed within the interval $[a, b]$.

So, the consequences derived in [14], mainly related to applications in error approximations, can be considered as optimal. Mainly, it treats on P_1-interpolation error estimate, the corrected trapezoidal quadrature formula, and finite element error approximations. For example, using the corrected trapezoidal quadrature formula, we have obtained an upper bound which is two times smaller than the errors obtained by using the classical trapezoidal quadrature formula. It highlights the importance and impact of the new Taylor-like formula (22)-(23) in assessing the accuracy of a given numerical approximation method.

This research can be extended to the case of dimension strictly greater than one. This extension requires a Taylor-like formula we have already derived in [12]. Additionally, we could expand this work to a second-order Taylor-like formula, as proposed in [13]. Both of these extensions will be explored to assess their impact on error estimates in the context of applications in numerical analysis.

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