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Longitudinal and transverse coherent waves in media containing randomly distributed spheres

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ABSTRACT

Multiple scattering effects due to a random distribution of identical spheres are investigated in the general case of elastic or poroelastic host media, where both longitudinal and transverse waves may co-exist. Propagation of plane coherent waves is assumed, and their dispersion equation looked for, as well as analytic approximations of those particular solutions that are close to the wavenumbers in the free host, when the product of the concentration with the scattering cross section of the spheres is low. Under this last condition, pair-correlation effects are seen to be of second order. Numerical studies are performed under the hole correction assumption, and compared to experimental data for tungsten carbide spheres in an epoxy matrix, which is a rather illustrative situation of how longitudinal and transverse waves participate to coherent propagation.

1. Introduction

Multiple scattering by random arrangements of scatterers is a topic with an extensive literature that has preoccupied researchers for many years, as can be seen, for example, from the many references in V.K. Varadan's et al. [1], Tsang et al.'s [2] and P. Martin's [3] books. The T-matrix method is well suited for the study of scattering by a two or three dimensional object, and there are basically two different approaches based on it in multiple scattering problems.

The first approach consists in writing down the multiple scattering equations and solving the resulting large linear system [4–6]. Its great advantage is the possibility of considering arbitrary dispersions of scatterers with any concentration, but its basic disadvantage is that the numerical work involved is computationally challenging, especially for three dimensional problems [7–9], even though recent and still ongoing research [10–12] makes the calculations faster for larger concentrations of particles. Effective properties may be deduced from such calculations, but the investigation of the effect on effective properties of different parameters such as the concentration of scatterers or their inner properties, is still cumbersome. Although T-matrix methods have been formulated for elastic wave scattering [13,14], very little work has been reported on simulating full elastic wave propagation in random particulate systems with mode conversion between longitudinal and rotational (transverse) waves [15].

In the second approach, knowledge of the statistics of the random distribution of scatterers is required and an average value of the scattered field is sought. Dispersion equations for the effective wavenumbers are looked for, and analytical

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formulas for a few approximate solutions can be obtained in some particular situations (low concentration, or large wavelengths to particles radius ratii, ..). This is a classic topic with a large literature (see *e.g.* Refs. [16,17]), with modern era dating from the works of Foldy [18], Lax [19] and Fikioris and Waterman [20]. This approach has the disadvantage of considering only isotropic random distributions of scatterers and of being limited in concentration, but has the advantage of producing analytical results that give insight into the scattering phenomena, and this is the reason why we chose to work within Fikioris and Waterman's framework. Again, elastic host media (vectorial case) have received less attention than the media where the propagation includes only one type of waves (scalar case).

Performing numerical calculations in order to analyze the propagation of elastic waves in a medium containing a random distribution of spheres will still be a challenge for the years to come. This is one of the reasons that explain the persistent interest of the statistical approach, which is at the heart of this paper. Not all methods are based on the T-matrix [21–24], but the latter has the advantage of dealing with both low and high frequencies and obtaining closed-form expressions for the wavenumbers of the coherent waves in all cases. More recently, the authors of Refs. [25,26], using the Wiener–Hopf technique, also with no assumptions on the wavelengths, the particle boundary conditions/size, or the volume fraction, have demonstrated the existence of several effective wavenumbers, even in the scalar case, but this is out of the topic of this paper.

The first paper, to our knowledge, dealing with the propagation of coherent waves in an elastic medium using the T-matrix is that of Varadan *et al.* [27], although a large part of the results had already been published in the book edited by V.K.Varadan and V.V.Varadan [1]. The most significant result was to obtain closed-form expressions for the longitudinal and transverse wavenumbers in the Rayleigh or low-frequency limit, and to show the excellent agreement between the phase velocity of the longitudinal coherent wave with Kinra's experimental data [28].

Neglecting longitudinal waves to keep only transverse waves amounts to considering the electromagnetic case, as treated in the 80's by Tsang *et* Kong [29] and then by Fikioris *et* Waterman [30] in a more comprehensive and clearer way. Unlike in Ref. [27], the extinction theorem is used in Ref. [30] in order to calculate the reflection and the transmission at the interface between a homogeneous medium and a multiple scattering one. This leads to the interesting result that the polarization of a transverse incident wave is preserved through the transmission process at the interface. This important point, which is not discussed by Fikioris *et* Waterman, can be deduced directly from the comparison of Eqs. (8,41,50) in Ref. [30]. As Varadan et al. [27], both Refs. [29,30] obtain a closed-form expression of the transverse wavenumber, but written explicitly in terms of the scattering coefficients of the T-matrix. The main advantage of a closed-form such as Eq. (57) in Ref. [29] is to be valid from low to high frequencies ; it is obtained from a first order asymptotic expansion in concentration, under the assumption of small spheres concentration, as in Foldy's approximation [18].

The elastic case has been considered for an incident longitudinal wave within the framework of the Quasi Crystalline Approximation in Ref. [31]. As in Ref. [29], closed-form expressions were given for all frequencies. The asymptotic expansions were performed up to order two in concentration, thus introducing the products between scattering coefficients and, consequently, the coupling between longitudinal and rotational waves (*c.f.* Eqs. ((29))–(32). In order to do so, all fields were assumed invariant with respect to the azimuth angle in the plane perpendicular to the direction of propagation of the coherent waves. This is debatable, especially for transverse coherent waves, and had led to a misuse of the scalar addition theorem all over, even for shear waves. No such assumption is done here, and Ref. [31] is corrected. The asymptotic expansions of the effective wavenumbers are also done around those in the free host, not only at low concentration as in Refs. [31,32], but in the more general case of low concentration times scattering, so that the effect of pair correlation between scatterers appears not only at third order, but also at second order.

We chose to follow the same steps as the statistical theory developed first in acoustics [20] and later in electromagnetism [29,30]: conditional ensemble averaging, under the quasi-crystalline approximation, of the basic multiple scattering equations that express the fields outside the particles in the host medium. These equations are obtained by means of the T-matrix that relates a scattered wave amplitude to an exciting one, and of an addition theorem that describes scattered waves from one particle as incident waves on another. Contrary to other more recent methods developed in the electromagnetic case in Refs. [33,34], the averaged exciting field on any given particle, or the effective field, is then supposed to be equal to a linear combination of plane waves propagating in the same direction as the incident plane wave that was at the origin of the multiple scattering process, and the dispersion equation that provides the effective wavenumbers is obtained therefrom. In elasticity, where both longitudinal and transverse waves propagate, both the scalar and the vectorial addition theorems have to be used, which had not been done in Ref. [31], but is here. We also consider any isotropic host medium, whether elastic [35], poroelastic [36,37] or else, as long as the T-matrix can be calculated. In short, the developed method generalizes the known results obtained in electromagnetism to elasticity, in particular for transverse incident waves which can generate rotational resonances [38], and also takes into account viscosity and thermal effects. It is also a corrected version of Ref. [31]. Applications of effective theories in such random media may range from ultrasonic characterization of suspensions [39] to metamaterials design [40], even for aqueous suspensions, where the shear waves in an even slightly viscous fluid can be of noticeable influence in attenuation measurements for example [41].

This paper is organized as follows: Section 2 deals with the description of the host medium that contains a random distribution of spheres. The average fields within Fikioris and Waterman's framewok are described in Section 3. The extinction theorem and the Lorentz–Lorenz law are obtained in Section 4. Section 5 is dedicated to the matrix form of the Lorentz–Lorenz law that leads to the dispersion equations of the coherent waves. Closed-form expressions of the effective wavenumbers are given in Section 6 when the product of the concentration with the scattering cross section is low. The particular case of elastic media is considered in Section 7, in which numerical results are compared to experimental data of Simon et al. [38,42].

2. Description of the host medium

We consider an isotropic medium in which L longitudinal (non rotational) waves and R rotational (divergence free) ones, may propagate. For example, $L = R = 1$ in a viscous fluid or in a (visco)elastic solid, $L = 2$ and $R = 1$ in a Biot medium or a heat conducting fluid, and $L = R = 2$ in a porous medium saturated with a viscous fluid [43]. This medium hosts, in the $z > 0$ region, a random array of identical spheres of radius a, and an incident plane wave propagating in the z direction gives rise to a multiple scattering process between all spheres; we look for the dispersion equation of the coherent waves that may describe the average propagation in the $z > 0$ region (see, for example, Ref. [44] for the definition of coherent waves).

We use the same methodology as in Refs. [31,45] and look again for asymptotic expansions of the effective wavenumbers, with no assumptions about the azimuth dependence of the fields, and using both the scalar addition theorem and the vectorial one to express the fields scattered from a sphere as incident waves upon another, contrary to what was done in Ref. [31].

The displacement field in the host medium is a linear combination of waves of all possible types,

$$\vec{u}\left(\vec{r}\right) = \sum_{p=1}^{L} \vec{\nabla}\phi^{(p)}\left(\vec{r}\right) + \sum_{p=1}^{R} \vec{\nabla} \wedge \vec{\Psi}_p\left(\vec{r}\right), \tag{1}$$

and, using Debye potentials [46], each potential vector $\vec{\Psi}_p$ is decomposed, in any local spherical orthornormal basis $\left\{\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi\right\}$, as $\vec{\Psi}_p\left(\vec{r}\right) = \vec{\nabla} \wedge (\phi^{(L+p)} r\vec{e}_r) + (1/k_p)\vec{\nabla} \wedge \vec{\nabla} \wedge (\phi^{(L+p+1)} r\vec{e}_r)$. The rotational waves that have a radial component of the displacement in a given spherical system may couple to longitudinal ones through scattering by a target centered at the center of that coordinate system; they will be termed as "s-waves", as in Ref. [46], and those that may not, as "t-waves". For a given rotational wave, there is one "s" and one "t" wave, associated to the same wavenumber, in a given local coordinate system ; change for another spherical coordinate system leads to a different mixture of the "s" and "t" parts of that same rotational wave.

Letting $P = L + 2R$ denote the number of different types of local polarization (one for each longitudinal wave, two for each rotational wave) in a given local spherical system, $\vec{u}^{(p)}$ will stand for the particle displacement associated to a wave of type p, with $p \in \mathcal{L} \Leftrightarrow 1 \leq p \leq L$ for a non-rotational wave, $p \in \mathcal{S} \Leftrightarrow p = L+1, L+3, \ldots, P-1$ for an "s" one, and $p \in \mathcal{T} \Leftrightarrow p = L+2, L+4, \ldots, P$ for a "t" one. p in $\mathcal{S}$ will be the "s" part of a rotational wave whose "t" part is $p+1$, and, of course, p in $\mathcal{T}$ will be the "t" part of a rotational wave whose "s" part is $p-1$, in a fixed given spherical system. Taking the well-known example of a Biot porous medium saturated with a fluid, $\vec{u}^{(1)}$ would be associated to the fast compressional wave, $\vec{u}^{(2)}$ to the slow one, $\vec{u}^{(3)}$ to the "s" part of the first rotational wave, and $\vec{u}^{(4)}$ to its "t" part. and $\vec{u}^{(5)}$, $\vec{u}^{(6)}$ to the "s" and "t" parts of a potential second rotational wave [43].

We use non normalized vector spherical harmonics and shorthand notations,

$$\vec{\mathrm{Z}}_{mn}^{(p)}(\vec{r}) = \begin{cases} \vec{\nabla}\mathrm{Z}_{mn}(k_p, \vec{r}) & \text{for} \quad p \in \mathcal{L}, \\ \vec{\nabla} \wedge \vec{\nabla} \wedge r\vec{e}_r \mathrm{Z}_{mn}(k_p, \vec{r}) & \text{for} \quad p \in \mathcal{S}, \\ \frac{1}{k_p}\vec{\nabla} \wedge \vec{\nabla} \wedge \vec{\nabla} \wedge r\vec{e}_r \mathrm{Z}_{mn}(k_p, \vec{r}) & \text{for} \quad p \in \mathcal{T}, \end{cases} \tag{2}$$

with $\mathrm{Z}_{mn} = \mathrm{J}_{mn}, \mathrm{H}_{mn}$, $k_{p+1} = k_p$ when $p \in \mathcal{S}$, and

$$\mathrm{J}_{mn}(k_p, \vec{r}) = \mathrm{j}_n(k_p r)\mathrm{P}_n^m\left(\cos\theta(\vec{r})\right)\mathrm{e}^{\mathrm{i}m\phi(\vec{r})}, \; \mathrm{H}_{mn}(k_p, \vec{r}) = \mathrm{h}_n^{(1)}(k_p r)\mathrm{P}_n^m\left(\cos\theta(\vec{r})\right)\mathrm{e}^{\mathrm{i}m\phi(\vec{r})}, \tag{3}$$

P_n^m the same associated Legendre function as in Ref. [47], and

$$\sum_{n,m} = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n}. \tag{4}$$

3. Average fields in Fikioris and Waterman's framework

3.1. The multiple scattering equations

Let $\vec{u}_{\mathrm{E}}^{(p)}(\vec{r}; \vec{r}_j)$ denote the displacement field of type p in the spherical coordinate system centered at $\vec{r}_j$, that, while observed at $\vec{r}$, excites a scatterer centered at $\vec{r}_j$. As in Ref. [31], we start with the integral equation, obtained after averaging the fields incident upon one scatterer over all possible locations of the others under the quasi-crystalline approximation. This integral equation governs the coherent fields, denoted by brackets, and states that the p-exciting wave $\vec{u}_{\mathrm{E}}^{(p)}(\vec{r}; \vec{r}_1)$ on a given target centered at $\vec{r}_1$ is due to the plane incident wave $\vec{u}_{\mathrm{inc}}^{(p)}(\vec{r}; \vec{r}_1)$ of the same type p, and to all other waves $\vec{u}_{\mathrm{E}}^{(q)}(\vec{r}; \vec{r}_j)$ of type q scattered by a sphere centered at $\vec{r}_j$ into waves of type u, thanks to the $T^{(qu)}(\vec{r}_j)$ scattering operator, provided those scattered waves represent, in the local system centered at $\vec{r}_1$, incident waves of type p:

$$\langle\vec{u}_{\mathrm{E}}^{(p)}(\vec{r}; \vec{r}_1)\rangle = \vec{u}_{\mathrm{inc}}^{(p)}(\vec{r}; \vec{r}_1) + \sum_{q=1}^{P}\sum_{u=1}^{P} \delta_{k_u k_p} \int \mathrm{d}\,\vec{r}_j\, \mathrm{n}(\vec{r}_j, \vec{r}_1)\mathrm{G}^{(up)}(\vec{r}_j, \vec{r}_1)T^{(qu)}(\vec{r}_j)\langle\vec{u}_{\mathrm{E}}^{(q)}(\vec{r}; \vec{r}_j)\rangle. \tag{5}$$

In Eq. (5), as well as in the following equations, except Eq. (24), letters as superscripts indicate a type of wave, longitudinal, s or t. Those of operators, fields, and matrices are in parentheses, while they are not for elements of the latter, in order to distinguish between them more easily. The $\mathrm{G}^{(up)}(\vec{r}_j,\vec{r}_1)$ and $T^{(qu)}(\vec{r}_j)$ operators are defined respectively in Eqs. (14) and (11). Transformation of a scattered u- wave in the spherical system of $\vec{r}_j$ into incident p- waves in the spherical system of $\vec{r}_1$ is possible only if k_p and k_u are equal, and it is illustrated in Eq. (5) by the $\delta_{k_u k_p}\mathrm{G}^{(up)}(\vec{r}_j,\vec{r}_1)$ operator defined in Eq. (14), while scattering of a q - wave into a u - wave at a sphere centered at $\vec{r}_j$ is illustrated by the $T^{(qu)}(\vec{r}_j)$ scattering operator defined below in Eqs. (11).

Eq. (5) is similar to Eq. (10) in Ref. [29] and Eq. (1) in Ref. [30], with longitudinal waves added, and the distinction between the "s" and "t" parts of a rotational wave in the spherical systems centered at $\vec{r}_j$ and $\vec{r}_1$ already done, while it was done only *after* decomposing the fields upon spherical harmonics in Refs. [29,30].

The integration in Eq. (5) is over the $(z > 0)$ region, $\mathrm{n}(\vec{r}_j,\vec{r}_1)$ is the conditional number density of spheres at $\vec{r}_j$ if one is known to be at $\vec{r}_1$, and we assume a constant density n_0 of scatterers of radius a, and conditional number density given by [48–50]

$$\mathrm{n}(\vec{r},\vec{r}_j)=\begin{cases} n_0\left[1+U\left(r,n_0\right)\right] & \text{for} \quad r=\left|\vec{r}-\vec{r}_j\right|>b,\\ 0 & \text{otherwise},\end{cases} \tag{6}$$

with $b \geq 2a$ and the Ursell function U obeying

$$\lim_{n_0\to 0} U(r,n_0)=\lim_{r\to\infty} U(r,n_0)=0. \tag{7}$$

In the following, harmonic wave motion is supposed with time dependence $\exp(-\mathrm{i}\omega t)$ understood.

The incident plane wave of amplitude $a^{(p)}$ propagates in the z direction ; in the spherical coordinates system centered on sphere number 1, it is either a linear combination of spherical harmonics of orders $m = 0$, if longitudinal, or $m = \pm 1$, if rotational; in that latter case, it is supposed in the following to be polarized in the y direction as in Ref. [29]:

$$\forall p \in \mathcal{L},$$
$$\vec{u}^{(p)}_{\mathrm{inc}}(\vec{r};\vec{r}_1)=a^{(p)}\mathrm{e}^{\mathrm{i}k_p z_1}\sum_{n=0}^{+\infty}\mathrm{i}^n(2n+1)\vec{\mathrm{J}}^{(p)}_{0n}\left(\vec{\rho}_1\right), \tag{8a}$$

$$\forall p \in \mathcal{S}$$
$$\begin{aligned}\vec{u}^{(p)}_{\mathrm{inc}}(\vec{r};\vec{r}_1)&=a^{(p)}\mathrm{e}^{\mathrm{i}k_p z_1}\sum_{n=1}^{+\infty}\mathrm{i}^n\frac{2n+1}{2}\left[\frac{1}{n(n+1)}\vec{\mathrm{J}}^{(p)}_{1n}\left(\vec{\rho}_1\right)+\vec{\mathrm{J}}^{(p)}_{-1n}\left(\vec{\rho}_1\right)\right],\\ \vec{u}^{(p+1)}_{\mathrm{inc}}(\vec{r};\vec{r}_1)&=a^{(p)}\mathrm{e}^{\mathrm{i}k_p z_1}\sum_{n=1}^{+\infty}\mathrm{i}^n\frac{2n+1}{2}\left[\frac{1}{n(n+1)}\vec{\mathrm{J}}^{(p+1)}_{1n}\left(\vec{\rho}_1\right)-\vec{\mathrm{J}}^{(p+1)}_{-1n}\left(\vec{\rho}_1\right)\right],\end{aligned} \tag{8b}$$

with z_j the z component of vector $\vec{r}_j$. The displacement fields are expressed as infinite series of vector spherical harmonics,

$$\langle\vec{u}^{(p)}_{\mathrm{E}}(\vec{r},\vec{r}_j)\rangle=\sum_{n,m}A^{(p)}_{mn}(\vec{r}_j)\vec{\mathrm{J}}^{(p)}_{mn}(\vec{\rho}_j) \quad \text{with} \quad \vec{\rho}_j\equiv\vec{r}-\vec{r}_j, \tag{9}$$

and, as there cannot be any monopolar rotational mode,

$$A^{(p)}_{m0}(\vec{r}_j)=0 \quad \text{if} \quad p\notin\mathcal{L}. \tag{10}$$

The action of the scattering operators $T^{(qp)}(\vec{r}_j)$ on a spherical harmonic is defined as

$$T^{(qp)}(\vec{r}_j)\vec{\mathrm{J}}^{(q)}_{mn}(\vec{\rho}_j)=T^{qp}_n\,\vec{\mathrm{H}}^{(p)}_{nm}(\vec{\rho}_j), \quad \text{with, after Eq. (10)}, \tag{11a}$$
$$T^{qp}_0=0 \quad \text{if} \quad (q,p)\notin\mathcal{L}^2, \tag{11b}$$

and, as "t" waves are uncoupled from the others [46],

$$T^{qu}_n=T^{uq}_n=0 \quad \text{if} \quad u\in\mathcal{T} \quad \text{while} \quad q\notin\mathcal{T}. \tag{12}$$

Eq. (5) can now be expressed with spherical harmonics, using Eqs. (9) and (11), as follows :

$$\begin{aligned}&\sum_{n,m}A^{(p)}_{mn}(\vec{r}_1)\vec{\mathrm{J}}^{(p)}_{mn}(\vec{\rho}_1)=\\ &\quad\vec{u}^{(p)}_{\mathrm{inc}}(\vec{r};\vec{r}_1)+\sum_{q=1}^{P}\sum_{u=1}^{P}\delta_{k_u k_p}\sum_{\nu,\mu}\int \mathrm{d}\,\vec{r}_j\,\mathrm{n}(\vec{r}_j,\vec{r}_1)A^{(q)}_{\mu\nu}(\vec{r}_j)T^{qu}_{\nu}\mathrm{G}^{(up)}\left(\vec{r}_j,\vec{r}_1\right)\vec{\mathrm{H}}^{(u)}_{\nu\mu}(\vec{\rho}_j).\end{aligned} \tag{13}$$

The action of operator $\mathrm{G}^{(up)}\left(\vec{r}_j, \vec{r}_1\right)$, that results from the addition theorem, is defined as follows

$$\mathrm{G}^{(up)}\left(\vec{r}_j, \vec{r}_1\right) \vec{\mathrm{H}}^{(u)}_{\nu\mu}(\vec{\rho}_j) = \sum_{n,m} \sum_{\ell} \mathrm{G}_{up}\left(n, m, \ell; \nu, \mu\right) \mathrm{e}^{\mathrm{i}(\mu-m)\phi(\vec{r}_{1j})} \mathrm{P}^{\mu-m}_{\ell}\left(\cos\theta(\vec{r}_{1j})\right) \mathrm{h}^{(1)}_{\ell}\left(k_p r_{1j}\right) \vec{\mathrm{J}}^{(p)}_{mn}\left(\vec{\rho}_1\right), \tag{14}$$

with $\vec{r}_{1j} = \vec{r}_1 - \vec{r}_j$.

The $\mathrm{G}_{up}\left(n, m, \ell; \nu, \mu\right)$ coefficients depend on the types of waves involved (longitudinal, "s" or "t"), and this is the reason why their *up* subscripts are identical to the superscripts in parentheses of the corresponding operator ; their expressions as well as a few of their properties are given in Appendix A. The values taken by ℓ in the sum over it depend on u, p, n, m, ν, μ, and [51,52] obey

$$\ell \equiv n + \nu \,(\mathrm{mod}\ 2) \quad \text{if} \quad u = p, \quad \text{and} \quad \ell \equiv n + \nu + 1 \,(\mathrm{mod}\ 2) \quad \text{else}. \tag{15}$$

Use of Eqs. (9) and (14)–(A.7) leads to

$$\begin{aligned} &\sum_{n,m} A^{(p)}_{mn}(\vec{r}_1) \vec{\mathrm{J}}^{(p)}_{mn}(\vec{\rho}_1) - \vec{u}^{(p)}_{\mathrm{inc}}(\vec{r}; \vec{r}_1) \\ &\quad = \sum_{n,m} \sum_{q=1}^{P} \sum_{u=1}^{P} \delta_{k_u k_p} \sum_{\nu,\mu} \sum_{\ell} T^{qu}_{\nu} \mathrm{G}_{up}\left(n, m, \ell; \nu, \mu\right) \vec{\mathrm{J}}^{(p)}_{mn}(\vec{\rho}_1) \\ &\quad \int \mathrm{d}\,\vec{r}_j\, \mathrm{n}(\vec{r}_j, \vec{r}_1) A^{(q)}_{\mu\nu}(\vec{r}_j) \mathrm{e}^{\mathrm{i}(\mu-m)\phi(\vec{r}_{1j})} \mathrm{P}^{\mu-m}_{\ell}\left(\cos\theta(\vec{r}_{1j})\right) \mathrm{h}^{(1)}_{\ell}\left(k_p r_{1j}\right). \end{aligned} \tag{16}$$

3.2. The coherent plane waves

The incident plane wave impinges the $z = 0$ interface at normal incidence and we expect the coherent waves of Eq. (9) they give rise to to be plane waves propagating and attenuated in the same direction z. The solutions of Eq. (16) are thus searched in the form [45]

$$A^{(q)}_{\mu\nu}(\vec{r}_j) = \sum_{s=1} \tilde{A}^{qs}_{\mu\nu} \mathrm{e}^{\mathrm{i}\xi_s z_j}, \tag{17}$$

so that any coherent wave of wavenumber ξ_s may arise from the combination of all possible types of waves in the host matrix. The summation over s extends *a priori* from 1 to infinity [25,26,53] ; as pointed out in Ref. [45], this summation is necessary for the coherent waves amplitudes not to be all equal to zero, but it is not mandatory when looking for the dispersion equation. More precisely, it is shown in Ref. [26] that a large number of coherent plane waves is required to calculate accurately the field in the vicinity of the interface between the host medium and the heterogeneous one, but that only one or two of them dominate away from that interface, at least in case of a fluid host matrix, as the others attenuate much more rapidly.

Noticing that the integration over $\vec{r}_j$ imposes $\mu = m$,

$$\begin{aligned} &\int \mathrm{d}\,\vec{r}_j\, \mathrm{n}(\vec{r}_j, \vec{r}_1) A^{(q)}_{\mu\nu}(\vec{r}_j) \mathrm{e}^{\mathrm{i}(\mu-m)\phi(\vec{r}_{1j})} \mathrm{P}^{\mu-m}_{\ell}\left(\cos\theta(\vec{r}_{1j})\right) \mathrm{h}^{(1)}_{\ell}\left(k_p r_{1j}\right) \\ &\qquad = (-1)^{\ell} \delta_{m,\mu} \sum_{s} I^{(p)}_{\ell}\left(\xi_s\right) \tilde{A}^{qs}_{m\nu}, \end{aligned} \tag{18}$$

with

$$I^{(p)}_{\ell}\left(\xi\right) = \int \mathrm{d}\,\vec{r}_j\, \mathrm{n}(\vec{r}_j, \vec{r}_1) \mathrm{e}^{\mathrm{i}\xi z_j} \mathrm{P}_{\ell}\left(\cos\theta(\vec{r}_{j1})\right) \mathrm{h}^{(1)}_{\ell}\left(k_p r_{j1}\right), \tag{19}$$

Eq. (16) turns to

$$\begin{aligned} &\sum_{s} \sum_{n,m} \tilde{A}^{ps}_{mn} \mathrm{e}^{\mathrm{i}\xi_s z_1} \vec{\mathrm{J}}^{(p)}_{nm}(\vec{\rho}_1) - \vec{u}^{(p)}_{\mathrm{inc}}(\vec{r}; \vec{r}_1) \\ &\quad = \sum_{s} \sum_{n,m} \sum_{q=1}^{P} \sum_{u=1}^{P} \delta_{k_u k_p} \sum_{\nu,m} \sum_{\ell} \tilde{A}^{qs}_{m\nu} T^{qu}_{\nu} (-1)^{\ell} I^{(p)}_{\ell}\left(\xi_s\right) \mathrm{G}_{up}\left(n, m, \ell; \nu, m\right) \vec{\mathrm{J}}^{(p)}_{nm}(\vec{\rho}_1). \end{aligned} \tag{20}$$

In the following, we shall, for a while, separate Eq. (20) into three different equations, depending upon the local polarization that p is related to, until we get the extinction theorem and compare it with those in Refs. [20,29] where a smaller number of polarization types of waves were considered.

Owing to the orthogonality of the vector spherical harmonics of different orders and/or degrees, and keeping in mind Eqs. (11b) and (A.1)–(A.7), this leads to

$$\forall p \in \mathcal{L}, \ \forall n \in \mathbb{N}, \ \forall m \in \mathbb{Z} \cap [-n, n]$$

$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - \delta_{0m} e^{ik_p z_1} a^{(p)} i^n (2n+1) =$$

$$\sum_s \sum_q \sum_{\nu=0}^{+\infty} \sum_\ell \tilde{A}^{qs}_{m\nu} T^{qp}_\nu (-1)^\ell I^{(p)}_\ell(\xi_s) G_{LL}(n,m,\ell;\nu,m), \tag{21a}$$

$\forall p \in \mathcal{S},\ \forall n \in \mathbb{N},\ \forall m \in \mathbb{Z} \cap [-n,n]$

$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - (1-\delta_{0n})\, i^n \frac{2n+1}{2} \left[\delta_{-1m} + \frac{1}{n(n+1)}\delta_{1m}\right] e^{ik_p z_1} a^{(p)} =$$

$$\sum_s \sum_q \sum_{\nu=1}^{+\infty} \sum_\ell \tilde{A}^{qs}_{m\nu} \left[T^{qp}_\nu G_{SS}(n,m,\ell;\nu,m) + T^{qp+1}_\nu G_{ST}(n,m,\ell;\nu,m)\right] (-1)^\ell I^{(p)}_\ell(\xi_s), \tag{21b}$$

$\forall p \in \mathcal{T},\ \forall n \in \mathbb{N},\ \forall m \in \mathbb{Z} \cap [-n,n]$

$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - (1-\delta_{0n})\, i^n \frac{2n+1}{2} \left[\delta_{-1m} + \frac{1}{n(n+1)}\delta_{1m}\right] e^{ik_p z_1} a^{(p)} =$$

$$\sum_s \sum_q \sum_{\nu=1}^{+\infty} \sum_\ell \tilde{A}^{qs}_{m\nu} \left[T^{qp}_\nu G_{SS}(n,m,\ell;\nu,m) + T^{qp-1}_\nu G_{ST}(n,m,\ell;\nu,m)\right] (-1)^\ell I^{(p)}_\ell(\xi_s). \tag{21c}$$

After Refs. [17,20,29,30], taking into account Eq. (6) provides

$$I^{(p)}_\ell(\xi) = \frac{2n_0 \pi i^\ell}{\xi - k_p} \left[\frac{2b}{\xi + k_p} \tilde{N}^{(p)}_\ell(\xi) e^{i\xi z_1} + \frac{i}{k_p^2} e^{ik_p z_1}\right] \text{with} \tag{22a}$$

$$\tilde{N}^{(p)}_\ell(\xi) = N^{(p)}_\ell(\xi) + \frac{\xi^2 - k_p^2}{k_p^3 b} L^{(p)}_\ell(\xi) \tag{22b}$$

$$N^{(p)}_\ell(\xi) = \xi b j'_\ell(\xi b) h^{(1)}_\ell(k_p b) - k_p b j_\ell(\xi_k b) h^{(1)'}_\ell(k_p b), \tag{22c}$$

$$L^{(p)}_\ell(\xi) = \int_b^\infty j_\ell(\xi r)\, h^{(1)}_\ell(k_p r)\, U(r, n_0)\, k_p^3 r^2 dr. \tag{22d}$$

Inserting Eqs. (22) into Eqs. (21) leads to

$\forall p \in \mathcal{L},\ \forall n \in \mathbb{N},\ \forall m \in \mathbb{Z} \cap [-n,n]$

$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - \delta_{0m} i^n (2n+1) e^{ik_p z_1} a^{(p)} =$$

$$\sum_s \frac{4n_0 \pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=0}^{+\infty} \sum_\ell i^\ell \tilde{A}^{qs}_{m\nu} T^{qp}_\nu (-1)^\ell \tilde{N}^{(p)}_\ell(\xi_s) G_{LL}(n,m,\ell;\nu,m)\, e^{i\xi_s z_1} +$$

$$\sum_s \frac{2n_0 \pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=0}^{+\infty} \sum_\ell (-1)^\ell i^{\ell+1} \tilde{A}^{qs}_{m\nu} T^{qp}_\nu G_{LL}(n,m,\ell;\nu,m)\, e^{ik_p z_1}, \tag{23a}$$

$\forall p \in \mathcal{S},\ \forall n \in \mathbb{N},\ \forall m \in \mathbb{Z} \cap [-n,n]$

$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - (1-\delta_{0n})\, i^n \frac{2n+1}{2} \left[\delta_{-1m} + \frac{1}{n(n+1)}\delta_{1m}\right] e^{ik_p z_1} a^{(p)} =$$

$$\sum_s \frac{4n_0 \pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \sum_\ell$$

$$i^\ell \tilde{A}^{qs}_{m\nu} \left[T^{qp}_\nu G_{SS}(n,m,\ell;\nu,m) + T^{qp+1}_\nu G_{ST}(n,m,\ell;\nu,m)\right] (-1)^\ell \tilde{N}^{(p)}_\ell(\xi_s) e^{i\xi_s z_1} +$$

$$\sum_s \frac{2n_0 \pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \sum_\ell$$

$$(-1)^\ell i^{\ell+1} \tilde{A}^{qs}_{m\nu} \left[T^{qp}_\nu G_{SS}(n,m,\ell;\nu,m) + T^{qp+1}_\nu G_{ST}(n,m,\ell;\nu,m)\right] e^{ik_p z_1}, \tag{23b}$$

$\forall p \in \mathcal{T},\ \forall n \in \mathbb{N},\ \forall m \in \mathbb{Z} \cap [-n,n]$

$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - (1-\delta_{0n})\, i^n \frac{2n+1}{2}\left[-\delta_{-1m} + \frac{1}{n(n+1)}\delta_{1m}\right] e^{ik_p z_1} a^{(p)} =$$
$$\sum_s \frac{4n_0\pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \sum_{\ell}$$
$$i^{\ell} \tilde{A}^{qs}_{m\nu}\left[T^{qp}_{\nu} \mathrm{G}_{SS}(n,m,\ell;\nu,m) + T^{qp-1}_{\nu} \mathrm{G}_{ST}(n,m,\ell;\nu,m)\right](-1)^{\ell} \tilde{N}^{(p)}_{\ell}(\xi_s) e^{i\xi_s z_1} +$$
$$\sum_s \frac{2n_0\pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \sum_{\ell}$$
$$(-1)^{\ell} i^{\ell+1} \tilde{A}^{qs}_{m\nu}\left[T^{qp}_{\nu} \mathrm{G}_{SS}(n,m,\ell;\nu,m) + T^{qp-1}_{\nu} \mathrm{G}_{ST}(n,m,\ell;\nu,m)\right] e^{ik_p z_1}. \tag{23c}$$

Obviously, if $m \neq 0, -1, 1$, all $\tilde{A}^{ps}_{mn}$ are zero, and, introducing

$$\gamma^{(m)}_0 = \eta^{(m)}_0 = 1 \tag{24a}$$

$$\gamma^{(m)}_n = \begin{cases} \dfrac{1}{i^n(2n+1)} & \text{if } m = 0 \\ \dfrac{2n(n+1)}{i^n(2n+1)} & \text{else,} \end{cases} \tag{24b}$$

$$\eta^{(m)}_n = \begin{cases} i^{-n} & \text{if } m = 0 \\ i^{-n} n(n+1) & \text{else,} \end{cases} \tag{24c}$$

and using Eqs. (10), (A.4), (A.7) and Eqs. (A.11), (A.12), (A.13), (A.16) of Appendix A, Eqs. (23) reduce to

$$\forall p \in \mathcal{L},\ \forall n \in \mathbb{N},\ \forall m \in [0,1],$$
$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - \frac{\delta_{0m}}{\gamma^{(m)}_n} e^{ik_p z_1} a^{(p)} =$$
$$\sum_s \frac{4n_0\pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=0}^{+\infty} \sum_{\ell} i^{\ell} \tilde{A}^{qs}_{m\nu} T^{qp}_{\nu} (-1)^{\ell} \tilde{N}^{(p)}_{\ell}(\xi_s) \mathrm{G}_{LL}(n,m,\ell;\nu,m)\, e^{i\xi_s z_1} +$$
$$\sum_s \frac{2n_0\pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=0}^{+\infty} i\delta_{0m} \frac{\eta^{(m)}_{\nu}}{\gamma^{(m)}_n} \tilde{A}^{qs}_{m\nu} T^{qp}_{\nu} e^{ik_p z_1}, \tag{25a}$$

$$\forall p \in \mathcal{S},\ \forall n \in \mathbb{N}^*,\ \forall m \in [0,1],$$
$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - \frac{\delta_{1m}}{\gamma^{(m)}_n} e^{ik_p z_1} a^{(p)} =$$
$$\sum_s \frac{4n_0\pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \sum_{\ell}$$
$$i^{\ell} \tilde{A}^{qs}_{m\nu}\left[T^{qp}_{\nu} \mathrm{G}_{SS}(n,m,\ell;\nu,m) + T^{qp+1}_{\nu} \mathrm{G}_{ST}(n,m,\ell;\nu,m)\right](-1)^{\ell} \tilde{N}^{(p)}_{\ell}(\xi_s) e^{i\xi_s z_1} +$$
$$\sum_s \frac{2n_0\pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} i \frac{\eta^{(m)}_{\nu}}{\gamma^{(m)}_n} \tilde{A}^{qs}_{m\nu} \delta_{1m} \left(T^{qp}_{\nu} + T^{qp+1}_{\nu}\right) e^{ik_p z_1}, \tag{25b}$$

$$\forall p \in \mathcal{T},\ \forall n \in \mathbb{N}^*,\ \forall m \in [0,1],$$
$$\sum_s \tilde{A}^{ps}_{mn} e^{i\xi_s z_1} - \frac{\delta_{1m}}{\gamma^{(m)}_n} e^{ik_p z_1} a^{(p)} =$$
$$\sum_s \frac{4n_0\pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \sum_{\ell}$$
$$i^{\ell} \tilde{A}^{qs}_{m\nu}\left[T^{qp}_{\nu} \mathrm{G}_{SS}(n,m,\ell;\nu,m) + T^{qp-1}_{\nu} \mathrm{G}_{ST}(n,m,\ell;\nu,m)\right](-1)^{\ell} \tilde{N}^{(p)}_{\ell}(\xi_s) e^{i\xi_s z_1} +$$
$$\sum_s \frac{2n_0\pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} i \frac{\eta^{(m)}_{\nu}}{\gamma^{(m)}_n} \tilde{A}^{qs}_{m\nu} \delta_{1m} \left(T^{qp}_{\nu} + T^{qp-1}_{\nu}\right) e^{ik_p z_1}. \tag{25c}$$

and, due to Eq. (A.11),

$$\tilde{A}^{ps}_{-1n} = \pm \frac{(n+1)!}{(n-1)!} \tilde{A}^{ps}_{1n}, \tag{26}$$

with the minus sign when $p \in \mathcal{T}$.

Because of Eq. (A.12) and as there is no coupling, through scattering, of "t" waves with either longitudinal or "s" waves, the coefficients of all $\tilde{A}^{qs}_{0n}$ $(m = 0)$ in Eqs. (25a) and (25b) are zero when $q \in \mathcal{T}$, and, for the same reasons, Eq. (25c) is a homogeneous linear system involving only p and q both in $\mathcal{T}$. When $m = 0$, thus, the $\tilde{A}^{ps}_{0n}$ are all zero when p is in $\mathcal{T}$, and "t" waves do not concur to the multiple scattering process in case of a longitudinal incident plane wave. The remaining $\tilde{A}^{ps}_{0n}$ unknowns obey Eqs. (25a) and (25b).

When $m = 1$, the incident plane wave is rotational, and, as Eq. (A.12) is no longer relevant, Eqs. (25) form one unique linear system that couples all the unknowns, and all types of waves participate to the multiple scattering process.

4. Extinction theorem and Lorentz–Lorenz law

The extinction theorem consists in balancing the coefficients of $\mathrm{e}^{\mathrm{i}k_p z_1}$ in Eqs. (25). It provides

$$\forall p \in \mathcal{L},$$

$$\mathrm{i} \sum_s \frac{2n_0\pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=0}^{+\infty} \eta_\nu^{(0)} \tilde{A}^{qs}_{0\nu} T^{qp}_\nu = -a^{(p)}, \tag{27a}$$

which is the extension of Eqs. (2.12-133) in Ref. [20] that takes into account the possibility of rotational waves to propagate in the matrix, and

$$\forall p \in \mathcal{S},$$

$$\mathrm{i} \sum_s \frac{2n_0\pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \mathrm{h}_\nu^{(1)} \tilde{A}^{qs}_{1\nu} \left(T^{qp}_\nu + T^{qp+1}_\nu\right) = -a^{(p)}, \tag{28a}$$

$$\forall p \in \mathcal{T},$$

$$\mathrm{i} \sum_s \frac{2n_0\pi}{(\xi_s - k_p)\, k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \mathrm{h}_\nu^{(1)} \tilde{A}^{qs}_{1\nu} \left(T^{qp}_\nu + T^{qp-1}_\nu\right) = -a^{(p)}, \tag{28b}$$

which are the extension of Eqs. (32,33) in Ref. [29] that takes into account the longitudinal waves. As $a^{(p)}$ is the same for $p \in \mathcal{S}$ as for $p \in T$, one can check that Eqs. (28) reduce to a single equation as in the electromagnetic case (Eq. (46) in Ref. [29]).

The Lorentz–Lorenz law consists in balancing the terms of $\mathrm{e}^{\mathrm{i}\xi_s z_1}$; there is one set of equations for $m = 0$, and a different one for $m = 1$, but both share the same general form,

$$\forall p \in \mathcal{L},\ \forall n \in \mathbb{N},\ \forall m \in [0, 1]\,,$$

$$\tilde{A}^{ps}_{mn} = \frac{4n_0\pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=0}^{+\infty} \sum_{\ell} \mathrm{i}^\ell \tilde{A}^{qs}_{m\nu} T^{qp}_\nu (-1)^\ell \tilde{N}^{(p)}_\ell(\xi_s) \mathrm{G}_{LL}\,(n, m, \ell; \nu, m)\,, \tag{29a}$$

$$\forall p \in \mathcal{S},\ \forall n \in \mathbb{N}^*,\ \forall m \in [0, 1]\,,$$

$$\tilde{A}^{ps}_{mn} = \frac{4n_0\pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \sum_{\ell} \mathrm{i}^\ell \tilde{A}^{qs}_{m\nu} \left[T^{qp}_\nu \mathrm{G}_{SS}\,(n, m, \ell; \nu, m) + T^{qp+1}_\nu \mathrm{G}_{ST}\,(n, m, \ell; \nu, m)\right] (-1)^\ell \tilde{N}^{(p)}_\ell(\xi_s), \tag{29b}$$

$$\forall p \in \mathcal{T},\ \forall n \in \mathbb{N}^*,\ \forall m \in [0, 1]\,,$$

$$\tilde{A}^{ps}_{mn} = \frac{4n_0\pi b}{\xi_s^2 - k_p^2} \sum_{q=1}^{P} \sum_{\nu=1}^{+\infty} \sum_{\ell} \mathrm{i}^\ell \tilde{A}^{qs}_{m\nu} \left[T^{qp}_\nu \mathrm{G}_{SS}\,(n, m, \ell; \nu, m) + T^{qp-1}_\nu \mathrm{G}_{ST}\,(n, m, \ell; \nu, m)\right] (-1)^\ell \tilde{N}^{(p)}_\ell(\xi_s). \tag{29c}$$

Setting to zero the determinant of the Lorentz–Lorenz law provides the dispersion equation the effective wavenumbers ξ_s must obey ; there is one dispersion equation for $m = 0$, and another one for $m = \pm 1$, and we should discuss that before proceeding any further.

The displacement field of a coherent wave propagating with a given effective wavenumber ξ_s, solution of the dispersion equation for $m = 0$, will be the summation of displacements of amplitudes $\tilde{A}^{ps}_{0n}$, $p \in \mathcal{L} \cup \mathcal{S}$. These displacements, in a given spherical coordinate system, will be linear combinations of spherical harmonics of order $m = 0$ (see Eqs. (9) and (17)), as is the case of a longitudinal plane wave (see Eq. (8a)) ; the coherent wave, which, in case of low concentration, should be close to a longitudinal wave, will be referred to, in the following, as a "quasi-longitudinal" coherent wave. If ξ_s is a solution of the dispersion equation for $m = 1$, the displacements, of amplitudes $\tilde{A}^{ps}_{\pm 1n}$, $p \in \mathcal{L} \cup \mathcal{S} \cup \mathcal{T}$, will be combinations of spherical harmonics of orders $m = \pm 1$, as is the case for a rotational wave (see Eq. (8b)). The coherent wave will be referred to as a "quasi-rotational" wave.

Solving anyone of those two dispersion equations consists in looking for the roots of an implicit equation in ξ_s, which is possible only numerically, and, as discussed in Refs. [25,26], while the number of solutions in each case is infinite, most of them correspond to highly attenuated waves. Moreover, provided a not too large concentration of scatterers, and/or not too much scattering from each, one expects coherent waves to propagate with wavenumbers close to those of the waves existing in the absence of scatterers. Approximations of those particular solutions of the dispersion equations are looked for in next section.

5. Matrix form of the Lorentz–Lorenz law : dispersion equations of the coherent waves

The aim of this section is to write Eqs. (29) in a matrix form more suitable to get approximate solutions of the dispersion equations, as in Refs. [31,45]. The quasi-longitudinal waves dispersion equation and the quasi-rotational one may be treated the same way until we get that matrix form, and, even though the elements of the matrices we shall introduce depend on the value (0 or 1) of m, we shall not write explicitly that they do, in order to have notations as light as possible. For this very same reason, we shall drop the index s all over and introduce new unknowns, $\tilde{B}^p_n$, that will correspond, for a given value of m and a given one of s, to $\tilde{A}^{ps}_{mn}$. Keeping in mind that there are no "s" or "t" monopolar modes, *i.e.* $n \in \mathbb{N}$ if $p \in \mathcal{L}$ and $n \in \mathbb{N}^*$ else, and turning back to the $\sum_u \delta_{k_u k_p}$ notations instead of expanding those sums so that the matrix form will be easier to get, we start with

$$\tilde{B}^p_n - \frac{4n_0\pi b}{\xi^2 - k_p^2} \sum_{u=1}^{P} \delta_{k_u k_p} \sum_{\nu=0}^{+\infty} \sum_{\ell} \tilde{N}^{(p)}_\ell(\xi)(-\mathrm{i})^\ell \mathrm{G}_{up}\,(n, m, \ell; \nu, m) \sum_{q=1}^{P} \tilde{B}^q_\nu T^{qu}_\nu = 0, \tag{30}$$

and introduce the $\mathbf{H}^{(pu)}(\xi)$ and $\mathbf{J}^{(pu)}(\xi)$ matrices from their (m-depending) elements,

$$\mathbf{H}^{pu}_{n\nu}(\xi) = \sum_{\ell} \mathrm{i}k_p b N^{(p)}_\ell(\xi)(-\mathrm{i})^\ell \mathrm{G}_{up}\,(n, m, \ell; \nu, m)\, \frac{\gamma^{(m)}_n}{\eta^{(m)}_\nu}, \tag{31a}$$

$$\mathbf{J}^{pu}_{n\nu}(\xi) = \sum_{\ell} (k_p b)^{-2} L^{(p)}_\ell(\xi)(-\mathrm{i})^\ell \mathrm{G}_{up}\,(n, m, \ell; \nu, m)\, \frac{\gamma^{(m)}_n}{\eta^{(m)}_\nu}. \tag{31b}$$

With $\gamma^{(m)}_n \tilde{B}^p_n$ denoted as B^p_n, Eq. (30) turns to

$$B^p_n - 4n_0\pi b^3 \sum_{u=1}^{P} \delta_{k_u k_p} \sum_{\nu=0}^{+\infty} \frac{1}{\mathrm{i}k_p b(\xi^2 - k_p^2) b^2} \left[\mathbf{H}^{pu}_{n\nu}(\xi) + \mathrm{i}(\xi^2 - k_p^2) b^2 \mathbf{J}^{pu}_{n\nu}(\xi)\right] \sum_{q=1}^{P} B^q_\nu \mathbf{t}^{(uq)}_{n\nu} = 0, \tag{32}$$

with matrix $\mathbf{t}^{(uq)}$ entries given by

$$\mathbf{t}^{(uq)}_{n\nu} = \delta_{\nu n} \frac{\eta_\nu}{\gamma_\nu} T^{qu}_\nu. \tag{33}$$

When multiple scattering is low, either because each single sphere does not scatter much the incident plane wave, or because there are very few of them, we expect each coherent quasi-longitudinal (resp. quasi-rotational) plane wave to be practically the same as one of the longitudinal (resp. rotational) plane wave in the pure matrix. Letting σ_{inc} be the scattering cross section associated to the incident plane wave of the spheres normalized by their geometric section πa^2, multiple scattering should be proportional somehow to σ_{inc} times the concentration c of spheres. For a longitudinal incident wave, $\sigma_{\mathrm{inc}} = \sum_{q=1}^{\mathrm{P}} \sigma_{\mathrm{inc}q}$ with $\sigma_{\mathrm{inc}q}$ given by [54]

$$\begin{cases} \sigma_{incq} = \dfrac{4}{(k_i nca)(k_q a)} \displaystyle\sum_{n=0}^{\infty} (2n+1) \left|T^{incq}_n\right|^2 \quad \text{if} \quad q \in \mathcal{L}, \\ \sigma_{incq} = \dfrac{4}{(k_i nca)(k_q a)} \displaystyle\sum_{n=1}^{\infty} n(n+1)(2n+1) \left|T^{incq}_n\right|^2 \quad \text{if} \quad q \in \mathcal{S}, \\ \sigma_{incq} = 0 \quad \text{else}, \end{cases} \tag{34}$$

and, for a rotational incident wave whose shear part corresponds to index p, $\sigma_{\text{inc}} = \sum_{q=1}^{P}\left(\sigma_{pq} + \sigma_{p+1q}\right)$ with $\sigma_{pq}, \sigma_{p+1q}$ given by [54]:

$$\begin{cases} \sigma_{pq} = \dfrac{2}{(k_p a)(k_q a)} \displaystyle\sum_{n=1}^{\infty} \dfrac{2n+1}{n(n+1)} \left|T_n^{pq}\right|^2 \quad \text{if} \quad q \in \mathcal{L}, \\ \sigma_{pq} = \dfrac{2}{(k_p a)(k_q a)} \displaystyle\sum_{n=1}^{\infty} (2n+1) \left|T_n^{pq}\right|^2 \quad \text{if} \quad q \in \mathcal{S}, \\ \sigma_{p+1q} = \dfrac{2}{(k_p a)(k_q a)} \displaystyle\sum_{n=1}^{\infty} (2n+1) \left|T_n^{p+1q}\right|^2 \quad \text{if} \quad q \in \mathcal{T}, \\ \sigma_{p+1q} = 0 \quad \text{if} \quad q \notin \mathcal{T}. \end{cases} \tag{35}$$

We define block matrix $\mathbb{T}$, whose $\mathbb{T}_{uq}$ entry, line u column q, is the $\mathbf{t}^{(uq)}$ matrix divided by σ_{inc}, along with

$$y_p = \left(\xi^2 - k_p^2\right) b^2, \qquad \epsilon = 4 n_0 \sigma_{inc} b^3 = \frac{3}{\pi} \sigma_{inc} \frac{b^3}{a^3} c, \tag{36}$$

so that small values of ϵ correspond to small concentration $\times$ scattering cross section products and ξ close to the wavenumber of the incident longitudinal or rotational plane wave. It is the approximation formulas of those particular values of ξ at small ϵ that, at the end, we are looking for.

Now we define the infinite vectors $|B^{(p)}\rangle$ and $|e\rangle$, of respective components B_n^p and $e_n = 1$, along with the infinite matrix $\bar{\mathbf{Q}}^{(pu)}(\xi)$,

$$\bar{\mathbf{Q}}_{n\nu}^{pu}(\xi) = \frac{\pi}{i k_p b y_p} \left[\mathbf{H}_{n\nu}^{pu}(\xi) + i y_p \mathbf{J}_{n\nu}^{pu}(\xi) - 1\right] \delta_{k_u k_p}, \tag{37}$$

and get

$$\left|B^{(p)}\right\rangle - \epsilon \sum_{u=1}^{P} \left(\bar{\mathbf{Q}}^{(pu)}(\xi) + \frac{\pi}{i k_p b y_p} \delta_{k_u k_p} \left|e\right\rangle \left\langle e\right|\right) \sum_{q=1}^{P} \mathbb{T}_{uq} \left|B^{(q)}\right\rangle = \left|0\right\rangle, \tag{38}$$

or

$$\left[\mathbb{I} - \epsilon \left(\bar{\mathbb{Q}}(\xi) + \sum_{p=1}^{P} \frac{\left|\mathrm{g}^{(p)}\right\rangle \left\langle e^{(p)}\right|}{y_p}\right) \mathbb{T}\right] \left|B\right\rangle = \left|0\right\rangle, \tag{39}$$

with the "block vectors" and block matrices defined from their entries,

$$(|B\rangle)_q = |B^{(q)}\rangle, \quad \left(|\mathrm{g}^{(p)}\rangle\right)_q = \delta_{pq}\sqrt{\pi}(i k_p b)^{-1/2}|e\rangle, \quad \left(\left\langle e^{(p)}\right|\right)_q = \delta_{k_q k_p}\sqrt{\pi}(i k_p b)^{-1/2} \left\langle e\right| \tag{40a}$$

$$(\mathbb{I})_{pq} = \delta_{pq} \left|e\right\rangle \left\langle e\right|, \quad \left(\bar{\mathbb{Q}}(\xi)\right)_{pq} = \bar{\mathbf{Q}}^{(pq)}(\xi). \tag{40b}$$

Now we can follow exactly Ref. [31], which followed merely the same procedure as Ref. [45] for its matrix formulation part; defining

$$|h^{(p)}\rangle = \mathbb{T}^{1/2}|\mathrm{g}^{(p)}\rangle, \quad \langle f^{(p)}| = \langle e^{(p)}|\mathbb{T}^{1/2}, \quad |b\rangle = \mathbb{T}^{1/2}|B\rangle, \tag{41a}$$

$$\mathbb{Q}(\xi) = \mathbb{T}^{1/2}\bar{\mathbb{Q}}(\xi)\mathbb{T}^{1/2}, \tag{41b}$$

and multiplying by $\mathbb{T}^{1/2}$ on the left, we obtain

$$\left[\mathbb{I} - \epsilon \left(\mathbb{Q}(\xi) + \sum_{p=1}^{P} \frac{|h^{(p)}\rangle\langle f^{(p)}|}{y_p}\right)\right] |b\rangle = |0\rangle, \tag{42}$$

the physical solutions ξ of which being the solutions of

$$\left[\mathbb{I} - \epsilon \left(\mathbb{I} - \epsilon\mathbb{Q}(\xi)\right)^{-1} \sum_{p=1}^{P} \frac{|h^{(p)}\rangle\langle f^{(p)}|}{y_p}\right] |b\rangle = |0\rangle. \tag{43}$$

Taking the inner product of Eq. (43) with $\langle f^{(p)}|$ yields

$$\left(1 - \epsilon \frac{M_{pp}(\xi)}{y_p}\right)\langle f^{(p)}|b\rangle - \epsilon \sum_{\substack{q=1 \\ q\neq p}}^{P} \frac{M_{pq}(\xi)}{y_q} \langle f^{(q)}|b\rangle = 0, \tag{44}$$

with

$$\forall p \in [1, P],$$
$$M_{pq}(\xi) = \langle f^{(p)}| \left(\mathbb{I} - \epsilon\mathbb{Q}(\xi)\right)^{-1} |h^{(q)}\rangle = \langle e^p|\mathbb{M}^{-1}\mathbb{T}|g^q\rangle. \tag{45}$$

and

$$\mathbb{M} = \left[\mathbb{I} - \epsilon\mathbb{T}\bar{\mathbb{Q}}(\xi)\right]^{-1} = \mathbb{I} + \epsilon\mathbb{T}\bar{\mathbb{Q}}(\xi) + \epsilon^2\left[\mathbb{T}\bar{\mathbb{Q}}(\xi)\right]^2 + \cdots \tag{46}$$

Eq. (44) is a homogeneous linear system of rank $P = L+2R$, whose determinant provides, when set to zero, the dispersion equation of the coherent waves. It is the generalization of Eqs. (48,49) obtained by Tsang *et* Kong [29] in electromagnetism.

6. The asymptotic effective wavenumbers at low concentration × scattering cross section product

At low ϵ, we expect each of the effective wavenumbers to be close to one k_p, so that we shall look for the asymptotic expansion in ϵ of each y_p for a given p,

$$y_p = \epsilon y_p^{(1)} + \epsilon^2 y_p^{(2)} + \cdots \tag{47}$$

It follows, from Eqs. (36), (45) and (46), that asymptotic expansions can also be done for the $\bar{\mathbb{Q}}(\xi)$ matrix, and hence for $M_{pq}(\xi)$,

$$\bar{\mathbb{Q}}(\xi) = \bar{\mathbb{Q}}(k_p) + \epsilon\bar{\mathbb{Q}}^{(1)} + \epsilon^2\bar{\mathbb{Q}}^{(2)} + \cdots, \quad M_{pq}(\xi) = M_{pq}^{(0)} + \epsilon M_{pq}^{(1)} + \epsilon^2 M_{pq}^{(2)} + \cdots. \tag{48}$$

Inserting Eqs. (47) and (48) into the determinant of Eq. (44), letting y_q be equal to $y_p + (k_p b)^2 - (k_q b)^2$ for all $q \neq p$ and considering only the first two terms of the asymptotic expansion of the effective wavenumbers ξ will provide approximations of those close to k_p at low concentration × scattering product.

We must study now separately the dispersion equation of the quasi-longitudinal coherent waves ($m = 0$) and that of the quasi-rotational waves ($m = 1$).

6.1. The effective wavenumbers of the quasi-longitudinal coherent waves

As noticed in the previous section, (locally) "t" waves do not participate to the multiple scattering process because of Eqs. (12) and (A.12), so that p is in $[1, L+R]$ rather than $[1, L+2R]$, the size of the homogeneous linear system Eq. (44), is $(L+R)\times(L+R)$, and $\delta_{k_u k_p} = \delta_{up}$. The dispersion equation takes a form similar to Eq. (27) in Ref. [31], and, taking into account the b^2 coefficient in the definition of ϵ in Eq. (36), as compared to that in Ref. [31], Eqs. (30) in the latter turn to

$$y_p^{(1)} = M_{pp}^{(0)}, \quad y_p^{(2)} = M_{pp}^{(1)} + \sum_{q\neq p} \frac{M_{pq}^{(0)} M_{qp}^{(0)}}{(k_p^2 - k_q^2)b^2}, \tag{49}$$

a priori whatever the value of the (L, R) couple. The expressions of the $M_{pp}^{(0)}, \quad M_{pp}^{(1)}$ in terms of the scattering coefficients, are given in Eqs. (B.6) and (B.7), and are to be compared to Eqs. (31) of Ref. [31].

Using Eqs. (B.1) and (B.3) and noticing that, contrary to what was the case in Ref. [32], the effect of correlation may be seen in $\bar{\mathbb{Q}}(k_p) = \bar{\mathbb{Q}}(\epsilon = 0)$, as the ϵ defined in Eq. (36) can be 0 while the concentration is not, one gets the asymptotic expansion of the ξ^2/k_p^2 ratio, up to order 2 in the concentration × scattering product,

$$\forall p \in \mathcal{L},$$
$$\frac{\xi^2}{k_p^2} = 1 - 3\mathrm{i}c\frac{\delta_1}{(k_p a)^3} - 9\mathrm{i}\frac{c^2}{(k_p a)^3}\left[\frac{b}{2a}\frac{\delta_2^{(p)}}{(k_p a)^2} + \frac{b}{a}\sum_{q\neq p}\frac{\delta_2^{(pq)}}{(k_p^2 - k_q^2)a^2} + \sum_q \frac{\delta_{2\mathrm{corr}}^{(pq)}}{(k_q a)^3}\right] \tag{50}$$

with, after Eqs. (B.4)–(B.7), (A.1),

$$\delta_1 = \sum_n (2n+1)T_n^{pp} \tag{51a}$$

$$\delta_2^{(p)} = \sum_n\sum_\nu \mathrm{i}^{\nu-n}(2\nu+1)T_\nu^{pp}\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{LL}(n, 0, \ell; \nu, 0)n_\ell(k_p b)T_n^{pp} \tag{51b}$$

$$\delta_2^{(pq)} = \sum_n\sum_\nu \mathrm{i}^{\nu-n}(2\nu+1)T_\nu^{pq}\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{qq}(n, 0, \ell; \nu, 0)N_\ell^{(q)}(k_p)T_n^{qp} \tag{51c}$$

$$\delta_{2\mathrm{corr}}^{(pq)} = \sum_n\sum_\nu \mathrm{i}^{\nu-n}(2\nu+1)T_\nu^{pq}\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{qq}(n, 0, \ell; \nu, 0)L_\ell^{(q)}(k_p)T_n^{qp}. \tag{51d}$$

The difference between the effective wavenumbers obtained from Eqs. (50) and (51a) and those from Ref. [31] will be studied in the case of elastic spheres in an elastic matrix in the numerical section.

6.2. The effective wavenumbers of the quasi-rotational coherent waves

The determinant to be set to zero is now of order $L+2R$ because of transverse waves, $y_{p+1}=y_p$ and, after Eqs. (45) and (B.1),

$$\forall p \in \mathcal{S},\ M_{p+1q}=M_{pq}, \qquad \forall (p,q)\in \mathcal{L}\times\mathcal{T},\ M^{(0)}_{pq}=0. \tag{52}$$

We look for effective wavenumbers close to a shear wavenumber of the matrix, *e.g.* for $p\in\mathcal{S}$ and get

$$y^{(1)}_p = M^{(0)}_{pp}+M^{(0)}_{pp+1},\quad y^{(2)}_p = M^{(1)}_{pp}+M^{(1)}_{pp+1}+\sum_{q\in\mathcal{L}}\frac{M^{(0)}_{pq}M^{(0)}_{qp}}{(k_p^2-k_q^2)b^2}+\sum_{\substack{q\in\mathcal{S},\\ q\neq p}}\frac{\left[M^{(0)}_{pq}+M^{(0)}_{pq+1}\right]\left[M^{(0)}_{qp}+M^{(0)}_{qp+1}\right]}{(k_p^2-k_q^2)b^2}, \tag{53}$$

and, after Eqs. (12), (B.10)–(B.13), (A.1)–(A.7),

$$\begin{aligned}&\forall p\in\mathcal{S},\\ &\frac{\xi^2}{k_p^2}=1-\frac{3}{2}ic\frac{\delta_1}{(k_pa)^3}-\frac{9i}{2}\frac{c^2}{(k_pa)^3}\times\\ &\left[\frac{b}{2a}\frac{\delta^{(p)}_2}{(k_pa)^2}+\frac{b}{a}\sum_{q\in\mathcal{L}}\frac{\delta^{(pq)}_{2L}}{(k_p^2-k_q^2)a^2}+\frac{b}{a}\sum_{\substack{q\in\mathcal{S}\\ q\neq p}}\frac{\delta^{(pq)}_{2S}}{(k_p^2-k_q^2)a^2}+\sum_{q\in\mathcal{L}}\frac{\delta^{(pq)}_{2\text{corrL}}}{(k_qa)^3}+\sum_{q\in\mathcal{S}}\frac{\delta^{(pq)}_{2\text{corrS}}}{(k_qa)^3}\right],\end{aligned} \tag{54}$$

with

$$\delta_1=\sum_{n^*}(2n+1)\left[T^{pp}_n+T^{p+1p+1}_n\right] \tag{55a}$$

$$\begin{aligned}\delta^{(p)}_2=&\sum_{n^*}\sum_{\nu^*}\mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}\quad\times\\ &\left\{\begin{aligned}&\left[T^{pp}_\nu T^{pp}_n+T^{p+1p+1}_\nu T^{p+1p+1}_n\right]\textstyle\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{SS}(n,1,\ell;\nu,1)n_\ell(k_pb)\\ &+\left[T^{p+1p+1}_\nu T^{pp}_n+T^{pp}_\nu T^{p+1p+1}_n\right]\textstyle\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{ST}(n,1,\ell;\nu,1)n_\ell(k_pb)\end{aligned}\right\}\end{aligned} \tag{55b}$$

$$\delta^{(pq)}_{2L}=\sum_{n^*}\sum_{\nu^*}\mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}T^{pq}_\nu\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{LL}(n,1,\ell;\nu,1)N^{(q)}_\ell(k_p)T^{qp}_n \tag{55c}$$

$$\begin{aligned}\delta^{(pq)}_{2S}=&\sum_{n^*}\sum_{\nu^*}\mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}\quad\times\\ &\left\{\begin{aligned}&\left[T^{pq}_\nu T^{qp}_n+T^{p+1q+1}_\nu T^{q+1p+1}_n\right]\textstyle\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{SS}(n,1,\ell;\nu,1)N^{(q)}_\ell(k_p)\\ &+\left[T^{pq}_\nu T^{q+1p+1}_n+T^{p+1q+1}_\nu T^{qp}_n\right]\textstyle\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{ST}(n,1,\ell;\nu,1)N^{(q)}_\ell(k_p)\end{aligned}\right\}\end{aligned} \tag{55d}$$

$$\delta^{(pq)}_{2\text{corrL}}=\sum_{n^*}\sum_{\nu^*}\mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}T^{pq}_\nu\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{LL}(n,1,\ell;\nu,1)L^{(q)}_\ell(k_p)T^{qp}_n \tag{55e}$$

$$\begin{aligned}\delta^{(pq)}_{2\text{corrS}}=&\sum_{n^*}\sum_{\nu^*}\mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}\quad\times\\ &\left\{\begin{aligned}&\left[T^{pq}_\nu T^{qp}_n+T^{p+1q+1}_\nu T^{q+1p+1}_n\right]\textstyle\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{SS}(n,1,\ell;\nu,1)L^{(q)}_\ell(k_p)\\ &+\left[T^{p+1q+1}_\nu T^{qp}_n+T^{pq}_\nu T^{q+1p+1}_n\right]\textstyle\sum_\ell(-\mathrm{i})^\ell \mathrm{G}_{ST}(n,1,\ell;\nu,1)L^{(q)}_\ell(k_p)\end{aligned}\right\}.\end{aligned} \tag{55f}$$

7. Elastic spheres in an elastic matrix : Numerics

In the following, we consider elastic spheres in an elastic matrix, so that one longitudinal ($L=1$) and one rotational wave ($R=1$) may propagate in the matrix, with respective wavenumbers $k_1=k_L$, $k_2=k_3=k_S$. In the following, indexes L will refer to longitudinal waves, while S and T will refer to "s" and "t" waves respectively. We suppose the Ursell function equal to 0, thus neglecting any pair correlation effect other than those of the hole correction, as we will be interested only in the effect of the (questionable) assumption of azimuthal invariance that was done in Ref. [31] when dealing with quasi-longitudinal waves, or of introducing longitudinal waves in Refs. [29,30] when dealing with quasi-rotational waves. The summations over n, ν in the Lorentz–Lorenz law, Eqs. (29), may be truncated to some finite integer N that depends

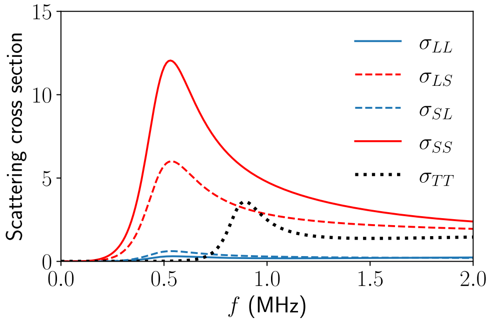

Fig. 1. Tungsten carbide sphere in an epoxy resin : scattering cross sections σ_{IM} for an incident wave of type I and a scattered wave of type M.

upon the frequency through the k_La, k_Sa products, while the number of unknowns depends upon the incident plane wave ($m = 0$ or $m = 1$), *i.e.* whether quasi-longitudinal waves or quasi-rotational ones are looked upon.

All figures correspond to an epoxy matrix and tungsten carbide spheres of radius $a = 198.5$ µm whose properties are given in Ref. [38], and the numerical calculations for $b = 2a$ are compared to previous experimental results [38,55], in which the longitudinal wave measurements were made in transmission at normal incidence in a water tank using a pair of immersion transducers with center frequency $f = 1$ MHz. The shear wave measurements were performed also in transmission at normal incidence using a contact measurement device. This device consisted of two blocks of aluminum alloy used as delay lines, on which shear wave transducers with center frequency $f = 1$ MHz were glued with controlled clamping. Each sample was inserted between the two delay lines and the coupling at the two interfaces was made by a shear wave couplant of controlled thickness [56,57].

Such a system exhibits two strong dipolar resonances, associated to translation and rotation movements [55,58] at frequencies respectively around 530 kHz and 920 kHz. In order to observe the influence of such resonances on the scattering by a single particle, the normalized scattering cross-sections [54] σ_{IM} for an incident wave of nature I and a scattered wave of nature M are plotted versus frequency in Fig. 1. The scattering cross section σ_{inc} of the incident wave is equal [54] to $\sigma_{LL} + \sigma_{LS}$ in case of an incident longitudinal wave, and to $\sigma_{SL} + \sigma_{SS} + \sigma_{TT}$ if the incident wave is rotational. The scattering cross sections σ_{LL}, σ_{LS}, σ_{SL}, and σ_{SS} exhibit a peak at the vicinity of the translation dipolar resonance of the bead, whereas σ_{TT} exhibits one in the vicinity of the rotation dipolar resonance. The translation resonance therefore affects both longitudinal and "s" waves, while the rotation resonance influences only the "t" waves. Moreover, for an incident longitudinal wave, the scattering cross section σ_{LS} is stronger than σ_{LL}, implying that the wave conversion is important, especially at the translation resonance. On the contrary, for an incident rotational wave, the scattering cross section σ_{SL} is weaker than σ_{SS}, implying that the wave conversion is weak in that case.

7.1. The quasi-longitudinal coherent waves

When looking for the quasi-longitudinal coherent waves, m is set to 0 in Eqs. (29), and, letting $A_\nu^{(L)}$ denote $\tilde{A}_{0\nu}^{1s}$ and $A_\nu^{(S)}$ denote $\tilde{A}_{0\nu}^{2s}$, remembering that $A_0^{(S)} = 0$, the dispersion equation is obtained by setting to zero the determinant of the following system of $(2N+1)$ equations of $2N+1$ unknowns $(A_0^{(L)}, \ldots, A_N^{(L)}, A_1^{(S)} \ldots A_N^{(S)})$,

$$\forall n \in [0, N]\,,\quad A_n^{(L)} = \frac{4n_0\pi b}{\xi^2 - k_L^2} \sum_{\nu=0}^{N} \sum_{\ell} (-\mathrm{i})^\ell \tilde{N}_\ell^{(L)}(\xi) \left[A_\nu^{(L)} T_\nu^{LL} + A_\nu^{(S)} T_\nu^{SL}\right] \mathrm{G}_{LL}(n, 0, \ell; \nu, 0)\,, \tag{56a}$$

$$\forall n \in [1, N]\,,\quad A_n^{(S)} = \frac{4n_0\pi b}{\xi^2 - k_S^2} \sum_{\nu=1}^{N} \sum_{\ell} (-\mathrm{i})^\ell \tilde{N}_\ell^{(S)}(\xi) \left[A_\nu^{(L)} T_\nu^{LS} + A_\nu^{(S)} T_\nu^{SS}\right] \mathrm{G}_{SS}(n, 0, \ell; \nu, 0)\,. \tag{56b}$$

At low concentration × scattering and under the hole correction assumption, the solution of Eq. (56) that is close to k_L can be approximated, using Eq. (50), by

$$\frac{\xi^2}{k_L^2} = 1 - 3\mathrm{i}c\frac{\delta_1}{(k_La)^3} - 9\mathrm{i}\frac{b}{a}\frac{c^2}{(k_La)^3}\left[\frac{\delta_2^{(L)}}{2(k_La)^2} + \frac{\delta_2^{(LS)}}{(k_L^2 - k_S^2)a^2}\right] \tag{57}$$

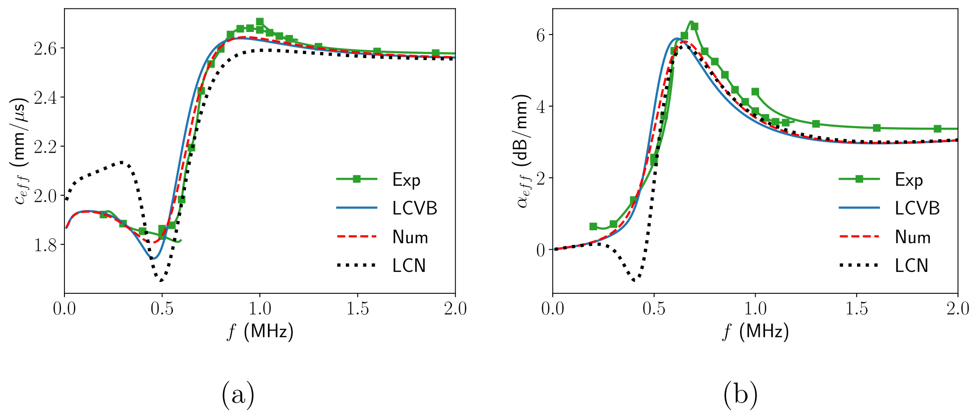

Fig. 2. Concentration $c = 5\%$ of tungsten carbide spheres in an epoxy matrix. (a) Phase velocity and (b) attenuation of the quasi-longitudinal coherent wave. Green line with squares: experiment from Ref. [55], black dotted line: from Eq. (36) of Ref. [31], blue solid line: from Eq. (57), dashed red line: from Eq. (56). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

with, after Eqs. (51a),

$$\delta_1 = \sum_{n=0}^{N} (2n+1) T_n^{LL} \tag{58a}$$

$$\delta_2^{(L)} = \sum_{n=0}^{N} \sum_{\nu=0}^{N} \mathrm{i}^{\nu-n} (2\nu+1) T_\nu^{LL} \sum_{\ell} (-\mathrm{i})^{\ell} \mathrm{G}_{LL}(n,0,\ell;\nu,0) n_\ell(k_L b) T_n^{LL} \tag{58b}$$

$$\delta_2^{(LS)} = \sum_{n=1}^{N} \sum_{\nu=1}^{N} \mathrm{i}^{\nu-n} (2\nu+1) T_\nu^{LS} \sum_{\ell} (-\mathrm{i})^{\ell} \mathrm{G}_{SS}\,(n,0,\ell;\nu,0)\, N_\ell^{(S)}(k_L) T_n^{SL}. \tag{58c}$$

The only difference between Eqs. (57) and (58) and Eqs. (36) of Ref. [31] where azimuthal invariance was assumed, comes from the $\delta_2^{(LS)}$ term in Eq. (58). It involves the $\mathrm{G}_{SS}\,(n,0,\ell;\nu,0)$ coefficient of the vector addition theorem, contrary to Eq. (36) which involved only the $\mathrm{G}_{LL}\,(n,0,\ell;\nu,0)$ coefficient of the scalar addition theorem. From a physical point of view, the coupling between longitudinal and shear waves was different in Ref. [31] from here. The first two delta coefficients in Eq. (58), associated to longitudinal waves only, are the same as in Ref. [31]. This suggests that the azimuthal invariance hypothesis is acceptable only if the coupling between longitudinal and shear waves is weak, as far as a longitudinal incident wave is considered.

The effective phase velocity $c_{eff} = \dfrac{2\pi f}{\mathrm{Re}[\xi]}$ and attenuation $\alpha_{eff} = \mathrm{Im}[\xi]$ of the quasi-longitudinal coherent wave are plotted in Figs. 2(a) and (b) respectively, for a concentration $c = 5\%$ of beads. Experimental results (green solid lines with squares) of Duranteau et al. [55] are compared to those obtained with Ref. [31] denoted LCN (black dotted curves) and those derived from Eqs. (57) denoted LCVB (blue solid curves). The red dashed curves labeled "Num" have been obtained by solving the dispersion equation that results from Eq. (56). For each frequency, this solution is obtained by searching for the complex effective wave number by a dichotomy method, in the vicinity of the value given by the approximate expression of Eq. (50). A convergence criterion is imposed during that search. The translation dipolar resonance of dense beads induces on coherent quasi-longitudinal wave a large dispersion of the phase velocity and a strong attenuation peak in the vicinity of 530 kHz, where the difference between the Num curve and the LCVB one is the largest, as $\sigma_{inc} = \sigma_{LL} + \sigma_{LS}$ and ϵ are larger in that region. The results of the modelings are quite comparable with Duranteau et al.'s [55] experimental data, except the LCN at low frequency: taking into account the azimuth variation of the fields has little influence at higher frequency.

7.2. The quasi-rotational coherent waves

When looking for the quasi-rotational coherent waves, m is set to 1 in Eqs. (29), and, letting $A_\nu^{(L)}$, $A_\nu^{(S)}$, and $A_\nu^{(T)}$ denote respectively $\tilde{A}_{1\nu}^{1s}$, $\tilde{A}_{1\nu}^{2s}$, $\tilde{A}_{1\nu}^{3s}$, remembering that $A_0^{(S)} = A_0^{(T)} = 0$, and that the summations over ℓ must obey Eq. (15), the dispersion equation is obtained by setting to zero the determinant of the following system of $(3N+1)$ equations of $3N+1$ unknowns,

$$\forall n \in [0, N]\,,$$
$$A_n^{(L)} =$$

$$\frac{4n_0\pi b}{\xi^2-k_L^2}\sum_{q=1}^{P}\sum_{\nu=0}^{N}\sum_{\ell}(-\mathrm{i})^{\ell}\left[A_\nu^{(L)}T_\nu^{LL}+A_\nu^{(S)}T_\nu^{SL}\right]\tilde{N}_\ell^{(L)}(\xi)\mathrm{G}_{LL}\left(n,1,\ell;\nu,1\right), \tag{59a}$$

$$\begin{aligned}&\forall n\in[1,N],\\ &A_n^{(S)}=\\ &\frac{4n_0\pi b}{\xi^2-k_S^2}\sum_{\nu=1}^{N}\sum_{\ell}(-\mathrm{i})^{\ell}\left[A_\nu^{(L)}T_\nu^{LS}+A_\nu^{(S)}T_\nu^{SS}\right]\mathrm{G}_{SS}\left(n,1,\ell;\nu,1\right)\tilde{N}_\ell^{(S)}(\xi)+\\ &\frac{4n_0\pi b}{\xi^2-k_S^2}\sum_{\nu=1}^{N}\sum_{\ell}(-\mathrm{i})^{\ell}A_\nu^{(T)}T_\nu^{TT}\mathrm{G}_{ST}\left(n,1,\ell;\nu,1\right)\tilde{N}_\ell^{(S)}(\xi),\end{aligned} \tag{59b}$$

$$\begin{aligned}&\forall n\,[1,N],\\ &A_n^{(T)}=\\ &\frac{4n_0\pi b}{\xi^2-k_S^2}\sum_{\nu=1}^{N}\sum_{\ell}(-\mathrm{i})^{\ell}\left[A_\nu^{(L)}T_\nu^{LS}\mathrm{G}_{ST}\left(n,1,\ell;\nu,1\right)+A_\nu^{(S)}T_\nu^{SS}\mathrm{G}_{ST}\left(n,1,\ell;\nu,1\right)\right]\tilde{N}_\ell^{(T)}(\xi)+\\ &\frac{4n_0\pi b}{\xi^2-k_S^2}\sum_{\nu=1}^{N}\sum_{\ell}(-\mathrm{i})^{\ell}A_\nu^{(T)}T_\nu^{TT}\mathrm{G}_{SS}\left(n,1,\ell;\nu,1\right)\tilde{N}_\ell^{(T)}(\xi).\end{aligned} \tag{59c}$$

At low concentration × scattering and under the hole correction assumption, the solution of Eq. (59) that is close to k_S can be approximated, using Eq. (54), by

$$\frac{\xi^2}{k_S^2}=1-\frac{3}{2}\mathrm{i}c\frac{\delta_1}{(k_Sa)^3}-\frac{9\mathrm{i}}{2}\frac{b}{a}\frac{c^2}{(k_Sa)^3}\left[\frac{1}{2}\frac{\delta_2^{(S)}}{(k_Sa)^2}+\frac{\delta_{2L}^{(SL)}}{(k_S^2-k_L^2)a^2}\right], \tag{60}$$

with, after Eqs. (55) and remembering Eq. (15),

$$\delta_1=\sum_{n=1}^{N}(2n+1)\left[T_n^{SS}+T_n^{TT}\right] \tag{61a}$$

$$\begin{aligned}\delta_2^{(S)}=&\sum_{n=1}^{N}\sum_{\nu=1}^{N}\mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}\quad\times\\ &\left\{\begin{array}{l}\left[T_\nu^{SS}T_n^{SS}+T_\nu^{TT}T_n^{TT}\right]\sum_\ell(-\mathrm{i})^\ell\mathrm{G}_{SS}(n,1,\ell;\nu,1)n_\ell(k_Sb)\\ +\left[T_\nu^{TT}T_n^{SS}+T_\nu^{SS}T_n^{TT}\right]\sum_\ell(-\mathrm{i})^\ell\mathrm{G}_{ST}(n,1,\ell;\nu,1)n_\ell(k_Sb)\end{array}\right\}\end{aligned} \tag{61b}$$

$$\delta_{2L}^{(SL)}=\sum_{n=1}^{N}\sum_{\nu=1}^{N}\mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}T_\nu^{SL}\sum_\ell(-\mathrm{i})^\ell\mathrm{G}_{LL}(n,1,\ell;\nu,1)N_\ell^{(L)}(k_S)T_n^{LS}. \tag{61c}$$

The first order term in Eqs. (60) is exactly the same as in Eq. (57) of Ref. [29]: the elastic and electromagnetic models are identical to first order in concentration times scattering cross section, and there is no coupling between shear and longitudinal waves at this order. Coupling occurs at higher orders.

The effective phase velocity and attenuation of the quasi-rotational coherent wave are plotted in Figs. 3(a) and (b) respectively, for a concentration $c = 6.4\%$ of beads. Experimental results (green curves with squares) of Simon et al. [38] are compared to those obtained with Refs. [29,30] denoted FW-TK (black dotted curves) and those derived from Eqs. (60) denoted LCVB (blue solid curves). The red dashed curves labeled "Num" have been obtained by solving the dispersion equation that results from Eq. (59). The same method was used as for the longitudinal coherent wave, except that the starting point of the dichotomy method used was given by Eq. (54). The translation and rotation dipolar resonances of the dense beads induce a large dispersion of the phase velocity and strong attenuation peaks in the vicinity of 530 kHz and 920 kHz. The difference between the Num curve and the LCVB one is larger in the vicinity of the rotation resonance frequency, where $\sigma_{inc}=\sigma_{SL}+\sigma_{ST}+\sigma_{TT}$ and hence ϵ are larger. The coupling between longitudinal and shear waves is also stronger at this resonance frequency, as shown by the difference between the FW-TK curve and the LCVB and/or Num curves. We still observe a good agreement of all models with Simon's experimental data [38].

8. Conclusion

Multiple scattering effects due to a random distribution of identical spheres have been investigated in the general case of elastic or poroelastic host media within Fikioris and Waterman's [20,30] framework. Setting to zero the determinants

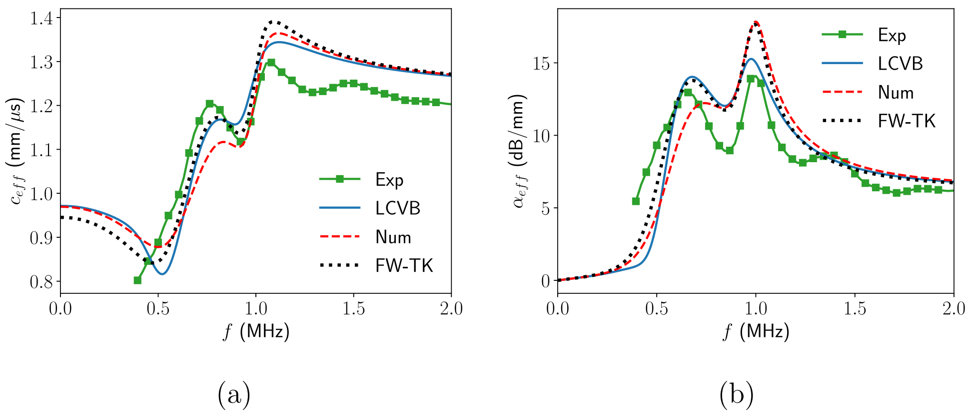

Fig. 3. Concentration $c = 6.4\%$ of tungsten carbide spheres in an epoxy matrix. (a) Phase velocity and (b) attenuation of the quasi-transverse coherent wave. Green line with squares: experiment, from Ref. [38], black dotted line: from Eq. (36) of Refs. [29,30], blue solid line: from Eqs. (60), red dashed line: from Eq. (59). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of Eqs. (29) and (44) provides the secular equations for the effective wavenumbers of the longitudinal and rotational coherent waves. Their closed form solutions, when the product of the concentration with the scattering cross section of the spheres is low, are given in Eqs. (50) and (51a) for the quasi-longitudinal coherent waves and in Eqs. (54) and (55) for the quasi-rotational ones. They correspond to asymptotic expansions up to order two in the concentration × scattering cross section product, introducing thus products between scattering coefficients and, consequently, the coupling between longitudinal and rotational waves.

In the case of elastic media, numerical studies have been performed and compared to experimental data for tungsten carbide spheres in an epoxy matrix. The results show a good agreement. Taking into account the azimuthal dependence of the fields has a notable influence on the propagation of longitudinal coherent waves only at low frequency.

For rotational coherent waves, the elastic and electromagnetic models provide the same asymptotic expansion up to first order in concentration × scattering cross section. There is no coupling between rotational and longitudinal waves at this order, and the first order term in Eqs. (60) and (61a) is exactly the same as in Eq. (57) of Ref. [29]. The major effect of the longitudinal - shear waves coupling is observed at the rotation resonance frequency of the spheres. There is a good agreement between the numerical solution obtained by setting to zero the determinant of Eq. (59), the closed form solutions Eqs. (60) and (61a) and experimental data from Ref. [38].

CRediT authorship contribution statement

Francine Luppé: Conceptualization, Methodology, Writing. **Jean-Marc Conoir:** Conceptualization, Methodology, Writing. **Tony Valier-Brasier:** Conceptualization, Methodology, Writing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix A. The addition theorems coefficients - Useful sums over ℓ

Different letters are often used for the coefficients of the addition theorems, such as [20,29,30,51,59,60] $C_{mn}^{\mu\nu}$ for the scalar addition theorem, $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$ for the vectorial one ; the G_{up} $(n, m, \ell; \nu, \mu)$ coefficients we use are related to those through the following equations.

- For u and p in $\mathcal{L}$,

$$\mathrm{G}_{up}\left(n,m,\ell;\nu,\mu\right)=\mathrm{G}_{LL}\left(n,m,\ell;\nu,\mu\right),\quad \text{with [51,29,59,30]}, \tag{A.1}$$

$$\sum_{\ell}\mathrm{G}_{LL}\left(n,m,\ell;\nu,\mu\right)\mathrm{e}^{\mathrm{i}(\mu-m)\phi_{j1}}\mathrm{P}_{\ell}^{\mu-m}\left(\cos\theta_{j1}\right)\mathrm{h}_{\ell}^{(1)}\left(k_p r_{j1}\right)=C_{mn}^{\mu\nu},\quad \text{and [51,30]} \tag{A.2}$$

$$\mathrm{G}_{LL}\left(n,m,\ell;\nu,\mu\right)=(-1)^m\mathrm{i}^{n-\nu+\ell}(2n+1)a(\mu,\nu|-m,n|\ell). \tag{A.3}$$

- For u and p in $\mathcal{S}$ or u and p in $\mathcal{T}$,

$$\mathrm{G}_{up}\left(n,m,\ell;\nu,\mu\right)=\mathrm{G}_{SS}\left(n,m,\ell;\nu,\mu\right),\quad \text{with [51,29,59,30]}, \tag{A.4}$$

$$\sum_{\ell}\mathrm{G}_{SS}\left(n,m,\ell;\nu,\mu\right)\mathrm{e}^{\mathrm{i}(\mu-m)\phi_{j1}}\mathrm{P}_{\ell}^{\mu-m}\left(\cos\theta_{j1}\right)\mathrm{h}_{\ell}^{(1)}\left(k_p r_{j1}\right)=A_{mn}^{\mu\nu},\quad \text{and [51,30]} \tag{A.5}$$

$$\mathrm{G}_{SS}\left(n,m,\ell;\nu,\mu\right)=\left(1-\delta_{n0}\right)\left(1-\delta_{\nu0}\right)(-1)^m a(\mu,\nu|-m,n|\ell)a(\nu,n,\ell). \tag{A.6}$$

- For p in $\mathcal{S}$ and u in $\mathcal{T}$,

$$\mathrm{G}_{up}\left(n,m,\ell;\nu,\mu\right)=\mathrm{G}_{pu}\left(n,m,\ell;\nu,\mu\right)=\mathrm{G}_{ST}\left(n,m,\ell;\nu,\mu\right),\quad \text{with [51,29,59,30]}, \tag{A.7}$$

$$\sum_{\ell}\mathrm{G}_{ST}\left(n,m,\ell;\nu,\mu\right)\mathrm{e}^{\mathrm{i}(\mu-m)\phi_{j1}}\mathrm{P}_{\ell}^{\mu-m}\left(\cos\theta_{j1}\right)\mathrm{h}_{\ell}^{(1)}\left(k_p r_{j1}\right)=B_{mn}^{\mu\nu},\quad \text{and [51,30]} \tag{A.8}$$

$$\mathrm{G}_{ST}\left(n,m,\ell;\nu,\mu\right)=\left(1-\delta_{n0}\right)\left(1-\delta_{\nu0}\right)(-1)^m a(\mu,\nu|-m,n|\ell,\ell-1)b(\nu,n,\ell). \tag{A.9}$$

The Gaunt coefficients $a(\mu,\nu|-m,n|\ell)$ are defined from [51]

$$\mathrm{P}_n^m\left(\cos\theta\right)\mathrm{P}_\nu^\mu\left(\cos\theta\right)=\sum_{\ell}a(m,n|\mu,\nu|\ell)\mathrm{P}_\ell^{m+\mu}\left(\cos\theta\right), \tag{A.10}$$

and the $a(\mu,\nu|-m,n|\ell,\ell-1)$, $a(\nu,n,\ell)$, $b(\nu,n,\ell)$, coefficients from Eqs. (13,14) in Ref. [30].

For $m=-1$, owing to the properties of the Gaunt coefficients, as noticed in Refs. [29,30],

$$\frac{\mathrm{G}_{LL}(n,-1,\ell;\nu,-1)}{\mathrm{G}_{LL}(n,1,\ell;\nu,1)}=\frac{\mathrm{G}_{SS}(n,-1,\ell;\nu,-1)}{\mathrm{G}_{SS}(n,1,\ell;\nu,1)}=-\frac{\mathrm{G}_{ST}(n,-1,\ell;\nu,-1)}{\mathrm{G}_{ST}(n,1,\ell;\nu,1)}$$
$$=\frac{(\nu-1)!\,(n+1)!}{(\nu+1)!\,(n-1)!}. \tag{A.11a}$$

After Ref. [52],

$$\mathrm{G}_{ST}\left(n,0,\ell;\nu,0\right)=0. \tag{A.12}$$

From Eqs. (15), (24), (A.1) and (A.10),

$$\sum_{\ell}(-1)^\ell\mathrm{i}^{\ell+1}\mathrm{G}_{LL}\left(n,m,\ell;\nu,m\right)=(2n+1)\mathrm{i}^{n-\nu+1}(-1)^m\sum_{\ell}a(m,\nu|m,n|\ell)=\mathrm{i}\delta_{0m}\frac{\eta_\nu^{(m)}}{\gamma_n^{(m)}}, \tag{A.13}$$

and, after Eq. (30) in Ref. [30],

$$\sum_{\ell}(-1)^\ell\mathrm{i}^{\ell+1}\mathrm{G}_{SS}\left(n,1,\ell;\nu,1\right)=-i\sum_{\ell}(-\mathrm{i})^\ell a(1,\nu|-1,n|\ell)a(\nu,n,\ell)=$$
$$i\frac{\eta_\nu^{(1)}}{\gamma_n^{(1)}}=\sum_{\ell}(-1)^\ell\mathrm{i}^{\ell+1}\mathrm{G}_{ST}\left(n,1,\ell;\nu,1\right), \tag{A.14a}$$

and Eq. (A.10) that provides

$$\sum_{\ell}a\left(0,\nu|0,n|\ell\right)=1, \tag{A.15}$$

along with Eqs. (82,13) in Ref. [30] and Eq. (15),

$$\sum_{\ell}\mathrm{i}^{\ell+1}\mathrm{G}_{SS}\left(n,0,\ell;\nu,0\right)=\sum_{\ell}(-\mathrm{i})^\ell\mathrm{G}_{SS}\left(n,0,\ell;\nu,0\right)=0. \tag{A.16}$$

Appendix B

The $M_{pq}^{(0)}$, $M_{pq}^{(1)}$ needed in the calculation of the $y_p^{(1)}$, $y_p^{(2)}$ of Eqs. (49) and (53) are obtained from the expansions of all functions of ξ around the k_p wavenumber it is supposed to be close to. In the following, r is no longer equal to the modulus of $\vec{r}$, but is a mute index related to a local polarization state.

$$M_{pq}^{(0)} = \frac{-i\pi}{\sigma_{inc}} \frac{1}{(k_p b)^{1/2}(k_q b)^{1/2}} \sum_r \delta_{k_r k_p} \sum_n \frac{\eta_n^{(m)}}{\gamma_n^{(m)}} T_n^{qr}, \tag{B.1}$$

$$M_{pq}^{(1)} = \frac{-i\pi}{\sigma_{inc}^2} \frac{1}{(k_p b)^{1/2}(k_q b)^{1/2}} \sum_r \sum_s \sum_u \sum_n \sum_\nu \frac{\eta_n^{(m)}\eta_\nu^{(m)}}{\gamma_n^{(m)}\gamma_\nu^{(m)}} T_\nu^{qs} \bar{\mathbf{Q}}_{n\nu}^{us}(k_p) T_n^{ur} \delta_{k_r k_p}, \tag{B.2}$$

or, using Eqs. (31) and (37),

$$\begin{aligned} M_{pq}^{(1)} = &-\frac{\pi^2}{\sigma_{inc}^2} \frac{\mathrm{i}}{(k_p b)^{1/2}(k_q b)^{1/2}} \sum_r \delta_{k_r k_p} \sum_u \sum_s \delta_{k_u k_s} \sum_n \sum_\nu \frac{\eta_n^{(m)}}{\gamma_\nu^{(m)}} T_\nu^{qs} T_n^{ur} \quad \times \\ &\left[(1-\delta_{k_u k_p}) \frac{h_{n\nu}^{(us)}(k_p)}{(k_p^2 - k_u^2) b^2} + \delta_{k_u k_p} \frac{q_{n\nu}^{(us)}(k_p)}{2(k_p b)^2} + \frac{j_{n\nu}^{(us)}(k_p)}{(k_u b)^3} \right] \\ &+ \frac{\pi^2}{\sigma_{inc}^2} \frac{1}{(k_p b)^{1/2}(k_q b)^{1/2}} \sum_r \delta_{k_r k_p} \sum_u \sum_s \delta_{k_u k_s} \sum_n \sum_\nu \frac{\eta_n^{(m)}\eta_\nu^{(m)}}{\gamma_n^{(m)}\gamma_\nu^{(m)}} (1-\delta_{k_u k_p}) \frac{T_\nu^{qs} T_n^{ur}}{k_u b (k_p^2 - k_u^2) b^2} \end{aligned} \tag{B.3a}$$

with

$$q_{n\nu}^{(us)}(k_p) = \sum_\ell (-\mathrm{i})^\ell \mathrm{G}_{su}(n, m, \ell; \nu, m) n_\ell(k_p b), \tag{B.4a}$$

$$n_\ell(x) = \mathrm{h}_\ell^{(1)}(x) \left\{ -x\mathrm{j}_\ell'(x) + \left[\ell(\ell+1) - x^2 \right] \mathrm{j}_\ell(x) \right\} - x^2 \mathrm{j}_\ell'(x) \mathrm{h}_\ell'(x), \tag{B.4b}$$

and

$$h_{n\nu}^{(us)}(k_p) = \frac{\mathbf{H}_{n\nu}^{us}(k_p)}{\mathrm{i}k_u b} \frac{\eta_\nu^{(m)}}{\gamma_n^{(m)}} = \sum_\ell (-\mathrm{i})^\ell \mathrm{G}_{su}(n, m, \ell; \nu, m) N_\ell^{(u)}(k_p), \tag{B.5a}$$

$$j_{n\nu}^{(us)}(k_p) = (k_u b)^2 \mathbf{J}_{n\nu}^{us}(k_p) \frac{\eta_\nu^{(m)}}{\gamma_n^{(m)}} = \sum_\ell (-\mathrm{i})^\ell \mathrm{G}_{su}(n, m, \ell; \nu, m) L_\ell^{(u)}(k_p). \tag{B.5b}$$

The expressions of all $M_{pq}^{(n)}$ thus depend upon the type of coherent waves involved (quasi-longitudinal or quasi-rotational), through m and the $\delta_{k_r k_p}$, and those needed in the asymptotic expressions of the effective wavenumbers, up to order 2 in ϵ, may be written as:

- for $m = 0$

$$M_{pq}^{(0)} = \frac{-\mathrm{i}\pi}{\sigma_{inc}} \frac{1}{(k_p b)^{1/2}(k_q b)^{1/2}} \sum_r \sum_n (2n+1) T_n^{qp}, \tag{B.6}$$

$$\begin{aligned} M_{pp}^{(1)} = &-\frac{\pi^2}{\sigma_{inc}^2} \frac{\mathrm{i}}{(k_p b)} \sum_q \sum_n \sum_\nu \mathrm{i}^{\nu-n} (2\nu+1) T_\nu^{pq} T_n^{qp} \quad \times \\ &\left[(1-\delta_{qp}) \frac{h_{n\nu}^{(qq)}(k_p)}{(k_p^2 - k_q^2) b^2} + \delta_{qp} \frac{q_{n\nu}^{(qq)}(k_p)}{2(k_p b)^2} + \frac{j_{n\nu}^{(qq)}(k_p)}{(k_q b)^3} \right] \\ &+ \frac{\pi^2}{\sigma_{inc}^2} \frac{1}{(k_p b)} \sum_{q \neq p} \sum_n \sum_\nu (2n+1)(2\nu+1) \frac{T_\nu^{pq} T_n^{qp}}{k_q b (k_p^2 - k_q^2) b^2} \end{aligned} \tag{B.7}$$

- for $m = 1$
  The $\delta_{k_r k_p}$ depend upon p, whether in $\mathcal{L}$ or not, and the M_{pq} needed for the expansion of the effective wavenumbers as well.

  - for $p \in \mathcal{L}$

$$M_{pq}^{(0)} = \frac{-\mathrm{i}\pi}{\sigma_{inc}} \frac{1}{2(k_p b)^{1/2}(k_q b)^{1/2}} \sum_n (2n+1) T_n^{qp}, \tag{B.8}$$

$$M_{pp}^{(1)} = -\frac{\pi^2}{\sigma_{inc}^2} \frac{\mathrm{i}}{2(k_p b)} \sum_q \sum_s \delta_{k_q k_s} \sum_n \sum_\nu \mathrm{i}^{\nu-n} (2\nu+1) \frac{n(n+1)}{\nu(\nu+1)} T_\nu^{ps} T_n^{qp} \quad \times$$

$$\left[(1-\delta_{qp})\frac{h_{n\nu}^{(qs)}(k_p)}{(k_p^2-k_q^2)b^2}+\delta_{qp}\frac{q_{n\nu}^{(qs)}(k_p)}{2(k_pb)^2}+\frac{j_{n\nu}^{(qs)}(k_p)}{(k_qb)^3}\right]$$
$$+\frac{\pi^2}{\sigma_{inc}^2}\frac{1}{4(k_pb)}\sum_{q\neq p}\sum_s\delta_{k_qk_s}\sum_n\sum_\nu(2n+1)(2\nu+1)\frac{T_\nu^{ps}T_n^{qp}}{k_qb(k_p^2-k_q^2)b^2} \tag{B.9a}$$

– for $p\in\mathcal{S}$

$$M_{pq}^{(0)}=\frac{-\mathrm{i}\pi}{\sigma_{inc}}\frac{1}{2(k_pb)^{1/2}(k_qb)^{1/2}}\sum_n(2n+1)\left(T_n^{qp}+T_n^{qp+1}\right), \tag{B.10}$$

$$M_{qp}^{(0)}=\frac{-\mathrm{i}\pi}{\sigma_{inc}}\frac{1}{2(k_pb)^{1/2}(k_qb)^{1/2}}\sum_\nu(2\nu+1)\left(T_\nu^{pq}+T_\nu^{pq+1}\right), \tag{B.11}$$

$$M_{pp}^{(1)}=-\frac{\pi^2}{\sigma_{inc}^2}\frac{\mathrm{i}}{2(k_pb)}\sum_q\sum_s\delta_{k_qk_s}\sum_n\sum_\nu \mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}T_\nu^{ps}\left(T_n^{qp}+T_n^{qp+1}\right)\quad\times$$
$$\left[(1-\delta_{k_qk_p})\frac{h_{n\nu}^{(qs)}(k_p)}{(k_p^2-k_q^2)b^2}+\delta_{k_qk_p}\frac{q_{n\nu}^{(qs)}(k_p)}{2(k_pb)^2}+\frac{j_{n\nu}^{(qs)}(k_p)}{(k_qb)^3}\right]$$
$$+\frac{\pi^2}{\sigma_{inc}^2}\frac{1}{4(k_pb)}\sum_q(1-\delta_{k_qk_p})\sum_s\delta_{k_qk_s}\sum_n\sum_\nu(2n+1)(2\nu+1)\frac{T_\nu^{ps}\left(T_n^{qp}+T_n^{qp+1}\right)}{k_qb(k_p^2-k_q^2)b^2} \tag{B.12a}$$

$$M_{pp+1}^{(1)}=-\frac{\pi^2}{\sigma_{inc}^2}\frac{\mathrm{i}}{2(k_pb)}\sum_q\sum_s\delta_{k_qk_s}\sum_n\sum_\nu \mathrm{i}^{\nu-n}(2\nu+1)\frac{n(n+1)}{\nu(\nu+1)}T_\nu^{p+1s}\left(T_n^{qp}+T_n^{qp+1}\right)\quad\times$$
$$\left[(1-\delta_{k_qk_p})\frac{h_{n\nu}^{(qs)}(k_p)}{(k_p^2-k_q^2)b^2}+\delta_{k_qk_p}\frac{q_{n\nu}^{(qs)}(k_p)}{2(k_pb)^2}+\frac{j_{n\nu}^{(qs)}(k_p)}{(k_qb)^3}\right]$$
$$+\frac{\pi^2}{\sigma_{inc}^2}\frac{1}{4(k_pb)}\sum_q(1-\delta_{k_qk_p})\sum_s\delta_{k_qk_s}\sum_n\sum_\nu(2n+1)(2\nu+1)\frac{T_\nu^{p+1s}\left(T_n^{qp}+T_n^{qp+1}\right)}{k_qb(k_p^2-k_q^2)b^2} \tag{B.13a}$$

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