Resonant acoustic scattering by two spherical bubbles

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The mutual interaction between two close bubbles in an acoustic field is studied. This interaction is modeled in the linear framework of the multiple scattering theory using spherical harmonics expansions and the addition theorem. In order to deal with small as well as large bubbles, viscous dissipation in the liquid, thermal dissipation in the gas, and surface tension are taken into account in the calculations of the scattering coefficients of a unique bubble. Under the assumption of the long wavelengths, the scattering coefficient of the monopolar mode is linked to the one obtained by using the Rayleigh-Plesset equation. The exact characteristic equation providing the symmetric and antisymmetric resonances of the two bubbles is established. Numerical results show that a great number of modes of vibration is required to describe the acoustic field around the bubbles. Moreover, whatever the spacing between two identical bubbles, the scattering cross section has a maximum value at the frequency of the symmetric mode while the antisymmetric mode is not detected. However, the strengthening of the scattering observed close to the symmetric resonance frequency is clearly due to the presence of the antisymmetric mode.
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I. INTRODUCTION

The scattering of acoustic waves by bubbles immersed in liquids or embedded in soft media is a fundamental problem, which has recently led to numerous applications from encapsulated microbubbles for acoustic imaging (Doinikov and Bouakaz, 2011) to bubble manipulations (Lanoy *et al.*, 2015) or seabed monitoring (Kubilius and Pedersen, 2016). Gas bubbles being very efficient resonators, interactions between them can be strong, leading to original properties of the effective acoustic waves (Commander and Prosperetti, 1989; Leroy *et al.*, 2008). In the case of two pulsating bubbles, interactions can induce Bjerknes forces, which attract or repel each other when bubbles oscillate in or out of phase, respectively (Ida, 2005; Pelekasis *et al.*, 2004). If many developments have been carried out on this topic so far (Barbat *et al.*, 1999; Doinikov, 2002; Doinikov and Zavtrak, 1997; Lauterborn and Kurz, 2010; Mettin *et al.*, 1997), some questions about the interaction between two bubbles in an acoustic field still remain, especially when bubbles are very closed.

A two-bubble system has two resonance frequencies, which correspond to oscillations of both bubbles in or out of phase. These resonance frequencies change as the bubbles approach each other (Feuillade, 1995). To describe the interaction between bubbles, most of the studies are based on the use of the Rayleigh-Plesset equation in the context of the hydrodynamic theory (Barbat and Ashgriz, 2004; Harkin *et al.*, 2001; Ida, 2002). With such an approach, a coupled oscillator system is used to describe the collective acoustic resonances from closely spaced bubbles in water. Because hydrodynamic models based on the Rayleigh-Plesset equation take into account only the monopole component in the interaction between bubbles, such an approach has a limited range of applicability and cannot be applied at distances compared with the size of the bubbles (Maksimov and Yusupov, 2016). Otherwise, Maksimov and Polovinka (2018) have originally modeled the interaction between closed bubbles using a bi-spherical coordinate system in the long wavelengths approximation. To be complete, the interaction between bubbles has also been derived in the linear framework of the multiple scattering neglecting viscous and thermal dissipation (Kapodistrias and Dahl, 2000; Skaropoulos *et al.*, 2003; Zheng and Apfel, 1995) or keeping only the monopolar modes (Feuillade, 2001). In this paper, the approach of the multiple scattering is adopted to tackle the problem of the interactions between very close bubbles. In order to deal with small as well as large bubbles in liquid, viscous dissipation in the liquid, thermal dissipation in the gas, and surface tension are taken into account. To achieve this, the multiple scattering theory we used is based on the ECAH model developed from the pioneering works of Epstein and Carhart (1953) and Allegra and Hawley (1972).

Interactions are modeled using spherical harmonics expansions and the addition theorem, and involve the scattering coefficients of each bubble. Such an approach allows to analyze the shift on the resonance frequencies, but also the damping coefficients, the scattering cross section, and the scattered pressure around the two-bubble system.

The paper is structured as follows. In Sec. II, the scattering coefficients by one bubble immersed in a viscous fluid with thermal effects in the bubble are obtained using the ECAH model. The multiple scattering by two bubbles is calculated in Sec. III in order to establish the exact characteristic equation providing the resonances. The resonant interactions between two bubbles are analyzed in Sec. IV. Section V deals

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with the analysis of the scattering cross sections. Section VI is the conclusion.

II. SCATTERING BY A GAS BUBBLE IMMERSED IN VISCOUS LIQUID

The aim is to calculate the scattering coefficients of a plane compressional wave by a unique gas bubble immersed in a viscous liquid. Before going further, note that the principles of fluid mechanics are used in Wang and Ye (2002) to derive a complete set of equations, which determines the acoustic scattering by a single gas filled bubble in water. The formulation is exact and takes into account the thermal and viscous effects in all vibrational modes. In addition, for the special case of low-frequency scattering, numerical results are shown to be in excellent agreement with Prosperetti's model at resonance (Prosperetti, 1984). Compared to the formulation developed in Wang and Ye (2002), our model differs by the three following points: (1) it is based on the ECAH acoustic model, but gives similar results if the same assumptions are used, (2) thermal effects in water and viscous effects in gas are neglected as in Prosperetti (1984), and (3) the surface tension at the surface of the bubble is taken into account only for the monopolar mode of vibration. Concerning point (2), we have assumed that the bubble wall temperature remains unperturbed, and this enable us to disregard the energy equation in the liquid, and the gas in the bubble is assumed perfect with the temperature nil at the gas–liquid interface. With regard to item (3), the surface tension comes from the use of the Laplace's law that has been linearized. Laplace's law applies to spherical bubbles; this is the reason why the surface tension is not applied to all vibrational modes but only to the monopolar mode. With our assumptions, we show that there is a full (analytical) correspondence between our results coming from the ECAH acoustic model and those of the hydrodynamic model (Prosperetti, 1984) for the monopolar mode at low frequency. This clearly shows that acoustic and hydrodynamic approaches lead to the same results provided that the same hypotheses are used. To our knowledge, this result has not yet been shown. To conclude this section, the scattering coefficients calculated in this chapter are those that are used later in the paper for studying the interaction between two bubbles.

A. Scattering amplitudes calculated from the ECAH model

The ECAH model has been widely used for studying colloids (Challis *et al.*, 2005) but it does not seem to have applications to bubbles. In our case, taking into account the thermal dissipation in the gas and the viscous dissipation in the liquid leads to assume the scattering of an entropic (thermal) wave in the bubble and a vorticity (shear) wave in the surrounding liquid.

The ambient liquid is characterized by the density ρ_0, adiabatic sound speed c_0, shear viscosity η_s, and bulk viscosity η_v. The longitudinal and transverse bulk wavenumbers in the liquid are defined by

$$k_L = \frac{\omega}{c_0\sqrt{1-i\omega\tau_v}} \quad \text{and} \quad k_T = \sqrt{\frac{i\rho_0\omega}{\eta_s}}, \tag{1}$$

where ω is the angular frequency, and

$$\tau_v = \frac{\eta_v + \frac{4}{3}\eta_s}{\rho_0 c_0^2} \tag{2}$$

is a characteristic viscous time in the liquid. The gas of the bubble is characterized by the density ρ_g, adiabatic sound speed c_g, thermal conductivity κ_p, thermal expansion coefficient α_p, heat coefficient at constant pressure per unit of mass C_p, and specific heat ratio γ_p. The longitudinal and entropic bulk wavenumbers in the gas are defined by (Bruneau *et al.*, 1989)

$$k_L^p = \frac{\omega}{c_g\sqrt{1-i\omega(\gamma-1)\tau_h}} \quad \text{and} \quad k_H^p = \sqrt{i\frac{\rho_g\omega C_p}{\kappa_p}}, \tag{3}$$

with

$$\tau_h = \frac{\kappa_p}{\rho_g c_g^2 C_p} \tag{4}$$

a characteristic thermal time in the gas.

For the sake of simplicity, the time dependence $\exp(-i\omega t)$ is omitted throughout the text. Moreover, the center of the considered bubble coincides with the origin of the coordinate system (O, r, θ, ϕ). As the incident longitudinal wave is assumed to propagate in the direction parallel to the z axis ($z = r\cos\theta$), the problem does not depend on the azimuthal angle ϕ. Scalar potentials associated to incident and scattered waves are thus expanded on spherical wave functions defined by

$$\begin{cases} \psi_n(k,\mathbf{r}) = j_n(kr)P_n(\cos\theta), \\ \chi_n(k,\mathbf{r}) = h_n(kr)P_n(\cos\theta), \end{cases} \tag{5}$$

where j_n are the spherical Bessel functions, h_n are the spherical Hankel functions of the first kind, and P_n are the Legendre polynomials. The scalar potential of the incident wave φ_I, those of scattered waves in the ambient liquid φ_M ($M = L,T$), and those corresponding to the fields in the bubble φ_M^p ($M = L,T$) can be expressed as

$$\begin{cases} \varphi_I(r,\theta) = \varphi_0\sum_{n=0}^{\infty}(2n+1)i^n\psi_n(k_L,\mathbf{r}), \\ \varphi_M(r,\theta) = \varphi_0\sum_{n=0}^{\infty}(2n+1)i^nT_n^M\chi_n(k_m,\mathbf{r}), \text{ with } M = L,T, \\ \varphi_M^p(r,\theta) = \varphi_0\sum_{n=0}^{\infty}(2n+1)i^nS_n^M\psi_n(k_m,\mathbf{r}), \text{ with } M = L,H, \end{cases} \tag{6}$$

where the scattering coefficients T_n^M ($M = L,T$) and S_n^M ($M = L,H$) are the unknown amplitudes calculated from the boundary conditions. The normal velocity u and the normal σ_{rr} and tangential $\sigma_{r\theta}$ stresses in the liquid at the bubble surface $r=R$ can therefore be expressed as follows:

$$\begin{cases} u(R,\theta) = \dfrac{\varphi_0}{R}\displaystyle\sum_{n=0}^{\infty} i^n(2n+1)\left[u_n^I + T_n^L u_n^L + T_n^T u_n^T\right]P_n(\cos\,\theta), \\ \sigma_{rr}(R,\theta) = \dfrac{\varphi_0}{R^2}\displaystyle\sum_{n=0}^{\infty} i^n(2n+1)\left[\sigma_n^I + T_n^L \sigma_n^L + T_n^T \sigma_n^T\right]P_n(\cos\,\theta), \\ \sigma_{r\theta}(R,\theta) = \dfrac{\varphi_0}{R^2}\displaystyle\sum_{n=1}^{\infty} i^n(2n+1)\left[\tau_n^I + T_n^L \tau_n^L + T_n^T \tau_n^T\right]\dfrac{\partial P_n(\cos\,\theta)}{\partial\theta}, \end{cases} \tag{7}$$

where the amplitudes u_n^M, σ_n^M, and τ_n^M (with $M=I,L,T$) are given by (Valier-Brasier *et al.*, 2015)

$$\begin{cases} u_n^I = nj_n(K_L) - K_L j_{n+1}(K_L), \\ u_n^L = nh_n(K_L) - K_L h_{n+1}(K_L), \\ u_n^T = n(n+1)h_n(K_T), \\ \sigma_n^I = -i\dfrac{\rho_0 c_0^2}{\omega}\left\{\left[(1-i\omega\tau_v)K_L^2 + 2in(n-1)\omega\tau_s\right]j_n(K_L) + 4i\omega\tau_s K_L j_{n+1}(K_L)\right\}, \\ \sigma_n^L = -i\dfrac{\rho_0 c_0^2}{\omega}\left\{\left[(1-i\omega\tau_v)K_L^2 + 2in(n-1)\omega\tau_s\right]h_n(K_L) + 4i\omega\tau_s K_L h_{n+1}(K_L)\right\}, \\ \sigma_n^T = 2n(n+1)\eta_s\left[(n-1)h_n(K_T) - K_T h_{n+1}(K_T)\right], \\ \tau_n^I = 2\eta_s\left[(n-1)j_n(K_L) - K_L j_{n+1}(K_L)\right], \\ \tau_n^L = 2\eta_s\left[(n-1)h_n(K_L) - K_L h_{n+1}(K_L)\right], \\ \tau_n^T = \eta_s\left[\left(2n^2-2-K_T^2\right)h_n(K_T) + 2K_T h_{n+1}(K_T)\right], \end{cases} \tag{8}$$

with $K_L = k_{LR}$, $K_T = k_{TR}$, and

$$\tau_s = \frac{\eta_s}{\rho_0 c_0^2}, \tag{9}$$

is another characteristic viscous time.

The radial velocity v, the acoustic pressure p, and the temperature field Θ in the bubble are given by (Allegra and Hawley, 1972; Epstein and Carhart, 1953)

$$\begin{cases} v(r,\theta) = \dfrac{\varphi_0}{r}\displaystyle\sum_{n=0}^{\infty}(2n+1)i^n\left[S_n^L v_n^L(r) + S_n^H v_n^H(r)\right]P_n(\cos\,\theta), \\ p(r,\theta) = \dfrac{\varphi_0}{r^2}\displaystyle\sum_{n=0}^{\infty}(2n+1)i^n\left[S_n^L p_n^L(r) + S_n^H p_n^H(r)\right]P_n(\cos\,\theta), \\ \Theta(r,\theta) = \dfrac{\varphi_0}{r^2}\displaystyle\sum_{n=0}^{\infty}(2n+1)i^n\left[S_n^L \Theta_n^L(r) + S_n^H \Theta_n^H(r)\right]P_n(\cos\,\theta), \end{cases} \tag{10}$$

with ($M=L,H$)

$$\begin{cases} v_n^M(r) = nj_n(k_m^p r) - k_m^p j_{n+1}(k_m^p r), \\ p_n^M(r) = i\rho_g\omega r^2 j_n(k_m^p r), \\ \Theta_n^M(r) = \dfrac{i}{\alpha_p\omega}\left[\gamma\left(\dfrac{\omega r}{c_g}\right)^2 - \left(k_m^p r\right)^2\right]j_n\left(k_m^p r\right). \end{cases} \tag{11}$$

At the interface $r = R$, the boundary conditions are the continuity of the radial velocities, the linearized Laplace's law, the vanishing of the shear stress and the temperature, namely

$$\begin{cases} u(R,\theta) = v(R,\theta), \\ \sigma_{rr}(R,\theta) = -p(R,\theta) - \dfrac{2i\sigma}{\omega R^2} u(R,\theta), \\ \sigma_{r\theta}(R,\theta) = 0, \\ \Theta(R,\theta) = 0, \end{cases} \tag{12}$$

where σ is the surface tension at the surface of the bubble. Then, using orthogonality properties of Legendre polynomials, this set of equations can be decoupled for each harmonic n. Laplace's law being only valid for perfectly spherical bubbles, we assume that the surface tension is only used for the monopolar mode $n = 0$, contrary to what is done in Wang and Ye (2002) where a more general theory is used.

The vanishing of the temperature field at the interface $r = R$ [last equation of Eq. (12)] allows to put the expression of the amplitude S_n^H in the following form:

$$S_n^H = \beta_n S_n^L \quad \text{with} \quad \beta_n = -\frac{\Theta_n^L(R)}{\Theta_n^H(R)}. \tag{13}$$

The amplitudes of the mode $n = 0$ are therefore solution of the matrix equation

$$\begin{pmatrix} T_0^L \\ S_0^L \end{pmatrix} = -\begin{pmatrix} u_0^L & -v_0^L - \beta_0 v_0^H \\ \sigma_0^L + \dfrac{2i\sigma}{\omega R} u_0^L & p_0^L + \beta_0 p_0^H \end{pmatrix}^{-1} \cdot \begin{pmatrix} u_0^I \\ \sigma_0^I + \dfrac{2i\sigma}{\omega R} u_0^I \end{pmatrix} \tag{14}$$

and those of the other modes ($n \neq 0$) are given by

$$\begin{pmatrix} T_n^L \\ T_n^T \\ S_n^L \end{pmatrix} = -\begin{pmatrix} u_n^L & u_n^T & -v_n^L - \beta_n v_n^H \\ \sigma_n^L & \sigma_n^T & p_n^L + \beta_n p_n^H \\ \tau_n^L & \tau_n^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} u_n^I \\ \sigma_n^I \\ \tau_n^I \end{pmatrix}. \tag{15}$$

B. Scattering coefficient of the monopolar mode in the long wavelength approximation

The scattering coefficients calculated from the ECAH model are valid for all frequencies. But, the comparison with the scattering coefficient obtained by Prosperetti *et al.* (1988) and Crum and Prosperetti (1983) concerns only the monopolar mode in the long wavelength approximation. In this case, Bessel and Hankel functions of argument K_L can be replaced by their asymptotic values. The normal velocity and normal stress in the liquid and the normal velocity and acoustic pressure in the bubble become

$$\begin{cases} u_0^L \approx \dfrac{i}{K_L}, \\ \sigma_0^L \approx -i\dfrac{\rho_0 c_0^2}{\omega}\left[(1 - i\omega\tau_v)(K_L^2 - iK_L) + \dfrac{4}{K_L}\omega\tau_s\right], \\ u_0^I = -\dfrac{K_L^2}{3}, \\ \sigma_0^I = -i\dfrac{\rho_0 c_0^2 K_L^2}{\omega}\left[1 - i\omega\left(\tau_v - \dfrac{4}{3}\tau_s\right)\right], \\ v_0^L + \beta_0 v_0^H \approx -\left(\dfrac{\omega R}{c_g}\right)^2 \dfrac{\gamma}{3\gamma}\left[1 - \dfrac{3(\gamma-1)}{(K_H^p)^2}\left(K_H^p \cot(K_H^p) - 1\right)\right], \\ p_0^L + \beta_0 p_0^H \approx i\rho_g \omega R^2. \end{cases} \tag{16}$$

It is worth noting that the argument K_H^p related to the entropic wave is not small, so that the asymptotic values of Bessel functions associated to K_H^p are not be used. As a consequence, the scattering coefficient T_0^L of the monopolar mode simplifies to

$$T_0^L = \frac{i\rho_0 c_0^2 K_L^3}{\dfrac{\rho_g c_g^2 \Phi}{\gamma} - \rho_0 c_0^2 (iK_L^3 + K_L^2) - 4i\omega\eta_s - \dfrac{2\sigma}{R}}, \tag{17}$$

with

$$\Phi = \frac{3\gamma (K_H^p)^2}{(K_H^p)^2 - 3(\gamma-1)\left[K_H^p \cot(K_H^p) - 1\right]}. \tag{18}$$

The parameter K_H^p being equal to $(i/\chi_g)^{1/2}$ where χ_g is the thermal diffusivity (Wang and Ye, 2002), this last expression is different from that of Prosperetti

$$\Phi = \frac{3\gamma (K_H^p)^2}{(K_H^p)^2 + 3(\gamma-1)\left[K_H^p \coth(K_H^p) - 1\right]} \tag{19}$$

only because the time dependence used ($e^{-i\omega t}$) is different. The expression of the scattering coefficient can also be written

$$T_0^L = -i\frac{\omega^2 K_L}{\omega^2 + 2i\omega(\Gamma_{\text{rad}} + \Gamma_{\text{th}} + \Gamma_{\text{vis}}) - \omega_0^2} \tag{20}$$

with

$$\begin{cases} \omega_0^2 = \dfrac{1}{\rho_0 R^2}\left[\dfrac{\rho_g c_g^2}{\gamma}\text{Re}(\Phi) - \dfrac{2\sigma}{R}\right], \\ \Gamma_{\text{rad}} = \dfrac{\omega_0^2 R}{2c_0}, \\ \Gamma_{\text{th}} = -\dfrac{\rho_g c_g^2}{2\gamma\rho_0\omega R^2}\text{Im}(\Phi), \\ \Gamma_{\text{vis}} = \dfrac{2\eta_s}{\rho_0 R^2}, \end{cases} \tag{21}$$

where the coefficients Γ_{rad}, Γ_{vis}, and Γ_{th} refer to the damping factors due to the acoustic radiation, viscous, and thermal effects, respectively (Leighton, 1994). This result is also in good agreement with the expressions given by the approach based on the Rayleigh-Plesset equation (Prosperetti, 1984; Prosperetti *et al.*, 1988). Nevertheless, Eq. (17) is a better approximation than Eq. (20) for which approximations have been performed in order to put the denominator as an oscillator system. The approximation consists in writing

$$K_L^3 = (\omega R/c_L)^3 \approx (\omega_0 R/c_L)^2 K_L . \tag{22}$$

In particular, Eq. (17) is in good agreement with numerical calculations of the monopolar mode until $k_0 R_0 \approx 0.5$ and a relative error inferior to 5% for small (until 1 μm) and large bubbles.

III. MODELING OF THE SCATTERING OF A PLANE COMPRESSIONAL WAVE BY TWO GAS BUBBLES

We are now interested by the scattering of a plane compressional wave by two spherical gas bubbles immersed in a viscous liquid and aligned with the direction of propagation. Bubbles radii are denoted R_1 and R_2, and the distance between bubble centers is denoted $2d$. If the two bubbles are far enough apart, shear waves are too attenuated to propagate from one to the other. This is the case if $\mathrm{Im}[k_T(2d - R_1 - R_2)] > 1$ (cf. Fig. 1), i.e., if the distance between bubble centers verifies the relation

$$2d > R_1 + R_2 + 2\sqrt{\frac{\eta_s}{\rho_0 \omega}}, \tag{23}$$

where the last term corresponds to two times the thickness of the viscous boundary layer of a unique bubble. According to this condition, the viscosity is well taken into account through the scattering coefficients calculated in Sec. I, but the propagation of shear waves between bubbles can be neglected. An exact solution was developed by Skaropoulos *et al.* (2003) for studying the acoustic multiple scattering from a cluster of bubbles in a fluid. It is inspired by the work of Gaunaurd *et al.* (1995), dealing with two soft or rigid spheres. In both cases, the method for obtaining the solution depends on the nature of the particle. We have modified the theory in order to get a solution valid whatever the kind of particle, drawing on the T matrix approach developed by Gumerov and Duraiswami (2005). More specifically, as our solution is expressed in terms of the scattering coefficients associated to one particle, changing the nature of the particle merely amounts to changing the scattering coefficients associated to one particle. To study the scattering of sound waves by two particles or many particles placed in various spatial configurations, the starting point is the addition theorem for the wave functions (Gumerov and Duraiswami, 2002).

A. Addition theorem

Figure 1 shows three coordinate systems with origins O, O_1, and O_2. The origins O_1 and O_2 coincide with the centers of both bubbles. The origin O is located on the line passing through points O_1 and O_2 at a distance d from them. The spherical coordinates of the coordinate system $\mathbb{R}_p$ centered at the point O_p are denoted (r_p,θ_p,ϕ_p). The axes being parallel, whatever the positions of O_1 and O_2, the scattered waves by each bubble do not depend on the azimuthal angles ϕ_1 and ϕ_2. The coordinate systems with origins O_1 and O_2 are, respectively, displaced by $\boldsymbol{r}_{01}$ and $\boldsymbol{r}_{02}$ from the primary coordinate system with origin O. Because their axes are parallel, there is no rotation of the coordinate systems. The translational addition theorem allows thus to express a spherical wave function of a primary coordinate system in terms of spherical waves functions in a secondary coordinate system (Lease and Thompson, 1991)

$$\begin{cases} \chi_n(k_L,\boldsymbol{r}_i) = \displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} \bar{G}(n,\nu,\mu)\chi_\mu(k_L,\boldsymbol{r}_{ij})\psi_\nu(k_L,\boldsymbol{r}_j), & \text{if } |\boldsymbol{r}_j| < |\boldsymbol{r}_{ij}|, \\ \chi_n(k_L,\boldsymbol{r}_i) = \displaystyle\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} \bar{G}(n,\nu,\mu)\psi_\mu(k_L,\boldsymbol{r}_{ij})\chi_\nu(k_L,\boldsymbol{r}_j), & \text{if } |\boldsymbol{r}_j| > |\boldsymbol{r}_{ij}|. \end{cases} \tag{24}$$

The coefficients $\bar{G}(n,\nu,\mu)$ are given by the relation

$$\bar{G}(n,\nu,\mu) = i^{\nu+\mu-n}(2\nu+1)G(0,n|0,\nu|\mu), \tag{25}$$

where the Gaunt coefficients $G(m,n|\mu,\nu|p)$ verify the relation (Lease and Thompson, 1991)

$$P_n^m(\cos\theta)P_\nu^\mu(\cos\theta) = \sum_{p=|n-\nu|}^{n+\nu} G(m,n|\mu,\nu|p)P_p^{m+\mu}(\cos\theta). \tag{26}$$

The values of the spherical coordinates of vectors $\boldsymbol{r}_{ij}$ are given in Table I.

B. Multiple interactions and the characteristic equation

The acoustic field is the sum of an incident wave φ_i and the scattered waves by both particles/bubbles φ_1 and φ_2. The potential associated to the incident plane wave is given by the spherical harmonics expansion

$$\varphi_i(\mathbf{r}_0) = \sum_{n=0}^{\infty}(2n+1)i^n\psi_n(k_L,\mathbf{r}_0). \tag{27}$$

Potentials associated to scattered waves by each particle are given as expansion on spherical wave functions χ_n

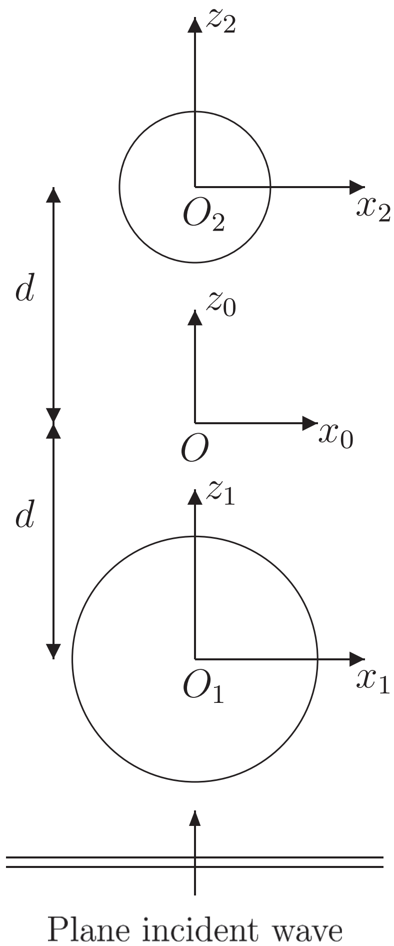

FIG. 1. Schema of two gas bubbles excited by a plane compressional wave.

$$\varphi_p(\mathbf{r}_p) = \sum_{n=0}^{\infty} B_n^{(p)} \chi_n(k_L, \mathbf{r}_p). \tag{28}$$

The incident wave on the particle p is the sum of the plane incident compressional wave and the scattered wave by the other particle

$$\varphi_{\text{inc}}^{(p)} = \sum_{n=0}^{\infty} \left[(2n+1) i^n \psi_n(\mathbf{r}_0) + B_n^{(q)} \chi_n(k_L, \mathbf{r}_q) \right], \tag{29}$$

with $q = 2$ if $p = 1$ and $q = 1$ if $p = 2$. Using the addition theorem, this potential can be calculated at the position $\boldsymbol{r}_p$ in the coordinate system $\mathbb{R}_p$, namely

$$\varphi_{\text{inc}}^{(p)}(\boldsymbol{r}_p) = \sum_{n=0}^{\infty} E_n^{(p)} \psi_n(k_L, \mathbf{r}_p), \tag{30}$$

with

$$E_n^{(p)} = \left[(2n+1) i^n e^{(-1)^p i k_L d} + \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} B_\nu^{(q)} \bar{G}(\nu, n, \mu) \chi_\mu(k_L, \boldsymbol{r}_{qp}) \right]. \tag{31}$$

Taking into account the scattering coefficients $T_n^{(p)}$ of the T matrix, which relates the expansion coefficients of the incident $E_n^{(p)}$ and scattered $B_n^{(p)}$ fields (Gumerov and Duraiswami, 2005), it follows from Eqs. (28) and (30) that we have

$$B_n^{(p)} = T_n^{(p)} \left[(2n+1) i^n e^{(-1)^p i k_L d} + \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} B_\nu^{(q)} \bar{G}(\nu, n, \mu) \chi_\mu(k_L, \boldsymbol{r}_{qp}) \right]. \tag{32}$$

Using this equation for both particles, we obtain the following matrix equation:

$$\begin{pmatrix} \mathcal{I} & \mathcal{Q}^{(1)} \\ \mathcal{Q}^{(2)} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}^{(1)} \\ \boldsymbol{B}^{(2)} \end{pmatrix} = \begin{pmatrix} e^{-ik_L d} \boldsymbol{A}^{(1)} \\ e^{ik_L d} \boldsymbol{A}^{(2)} \end{pmatrix}, \tag{33}$$

where $\mathcal{I}$ is the identity matrix. In this equation, the components of the vector $\boldsymbol{B}^{(p)}$ are the scattering amplitudes $B_n^{(p)}$ of both particles taking into account the multiple interactions, and the components of the vector $\boldsymbol{A}^{(p)}$ are the scattering amplitudes of each particle without interaction given by

$$A_n^{(p)} = (2n+1) i^n T_n^{(p)}. \tag{34}$$

The components of the block matrices $\mathcal{Q}^{(p)}$ are defined by the relation

$$Q_{n\nu}^{(p)} = T_n^{(p)} \sum_{\mu=0}^{\infty} \bar{G}(n, \nu, \mu) \chi_\mu(k_L, \boldsymbol{r}_{qp}), \tag{35}$$

with $q = 2$ if $p = 1$ and $q = 1$ if $p = 2$. Inserting the coefficients $B_n^{(p)}$ in Eq. (28) leads to calculate the acoustic field scattered by the two bubbles.

If the source term represented by the incident plane wave is canceled, the matrix equation (33) becomes homogeneous and the resonances of the bubble-bubble system are given by the characteristic equation

$$\det\left[\mathcal{I} - \mathcal{Q}^{(1)} \mathcal{Q}^{(2)}\right] = 0, \tag{36}$$

which is an exact equation obtained without approximation. It can be noted that the exact characteristic equation contains all the vibration modes so that the matrices in Eq. (36) are of infinite dimension. It should be noted that all the numerical results of the paper are obtained by solving the characteristic equation (36).

TABLE I. Values of spherical coordinates of vectors $\boldsymbol{r}_{ij}$.

$\boldsymbol{r}_{ij}$	$\boldsymbol{r}_{01}$	$\boldsymbol{r}_{10}$	$\boldsymbol{r}_{02}$	$\boldsymbol{r}_{20}$	$\boldsymbol{r}_{12}$	$\boldsymbol{r}_{21}$
$\lvert\boldsymbol{r}_{ij}\rvert$	d	d	d	d	$2d$	$2d$
θ_{ij}	π	0	0	π	0	π

C. Approximated characteristic equation at long wavelength

We have assumed that $2d > R = \sup(R_1, R_2)$ [cf. Eq.(23)], but now let us see what happens if $2d \gg R$. In this case, bubbles are supposed to remain spherical, and it is commonly accepted that the use of the monopolar mode of vibration $n=0$ is relevant to describe the bubble dynamics and their interactions. The goal here is to show that the acoustic model allows to retrieve the results obtained at low frequency when $2d \gg R$.

In this section, we neglect all the vibration modes $n \neq 0$ and the matrix equation (33) reduces to

$$\begin{pmatrix} 1 & -T_0^{(1)} h_0(2k_L d) \\ -T_0^{(2)} h_0(2k_L d) & 1 \end{pmatrix} \begin{pmatrix} B_0^{(1)} \\ B_0^{(2)} \end{pmatrix} = \begin{pmatrix} T_0^{(1)} e^{-jk_L d} \\ T_0^{(2)} e^{jk_L d} \end{pmatrix}, \tag{37}$$

leading to the following expressions of the scattering amplitudes:

$$\begin{cases} B_0^{(1)} = T_0^{(1)} \dfrac{e^{-jk_L d} + T_0^{(2)} h_0(2k_L d) e^{jk_L d}}{1 - T_0^{(1)} T_0^{(2)} [h_0(2k_L d)]^2}, \\ B_0^{(2)} = T_0^{(2)} \dfrac{e^{jk_L d} + T_0^{(1)} h_0(2k_L d) e^{-jk_L d}}{1 - T_0^{(1)} T_0^{(2)} [h_0(2k_L d)]^2}. \end{cases} \tag{38}$$

The resonances of the two-bubble system can be calculated from Eq. (36) but correspond also to frequencies for which the denominator of the scattering coefficients in Eq. (38) vanishes, namely

$$1 - T_0^{(1)} T_0^{(2)} [h_0(2k_L d)]^2 = 0. \tag{39}$$

This equation looks like the one of a two-mirror Fabry-Perot resonator of geometrical length $2d$ where the bubbles play the role of the mirrors with reflection coefficients $T_0^{(1)}$ and $T_0^{(2)}$. If the distance between bubbles is smaller than the wavelength, the expansion at the first order of the Hankel function $h_0(2k_L d) \approx -i/(2k_L d)$ can be used. In addition, we have shown, cf. Eq. (17), that the scattering coefficient $T_0^{(p)}$ in the long wavelength regime can be approximated by

$$T_0^{(p)} = -i \frac{\omega^2 k_L R_p}{\omega^2 + 2i\omega\Gamma_p - \omega_{p_0}^2} \tag{40}$$

with

$$\Gamma_p = \Gamma_{\mathrm{rad}}^p + \Gamma_{\mathrm{th}}^p + \Gamma_{\mathrm{vis}}^p = \frac{\omega_{p_0}^2 R_p}{2c_0} - \frac{\rho_g c_g^2}{2\gamma \rho_0 \omega R_p^2} \mathrm{Im}(\phi_p) + \frac{2\eta_s}{\rho_0 R_p^2}, \tag{41}$$

where Φ_p and $\omega_{p_0} = 2\pi f_{p_0}$ are given by Eqs. (18) and (21), respectively. Under these assumptions, the characteristic equation (39) can finally be written as follows:

$$\omega^4 \left[1 - \frac{R_1 R_2}{4d^2}\right] + 2i\omega^3 (\Gamma_1 + \Gamma_2) - \omega^2 \left(\omega_{1_0}^2 + \omega_{2_0}^2 + 4\Gamma_1\Gamma_2\right) - 2i\omega\left(\Gamma_2 \omega_{1_0}^2 + \Gamma_1 \omega_{2_0}^2\right) + \omega_{1_0}^2 \omega_{2_0}^2 = 0. \tag{42}$$

This equation is a generalization of Eq. (16) given by Ida (2002) when the damping coefficients Γ_1 and Γ_2 are neglected. In this last case, the previous equation is reduced to

$$\omega^4 \left[1 - \frac{R_1 R_2}{4d^2}\right] - \omega^2 \left(\omega_{1_0}^2 + \omega_{2_0}^2\right) + \omega_{1_0}^2 \omega_{2_0}^2 = 0, \tag{43}$$

and the resonance frequencies of the two-bubble system are then given by

$$\begin{cases} \left(\omega_+^{\mathrm{th}}\right)^2 = \dfrac{\omega_{1_0}^2 + \omega_{2_0}^2 - \sqrt{\left(\omega_{1_0}^2 - \omega_{2_0}^2\right)^2 + 4\omega_{1_0}^2 \omega_{2_0}^2 \dfrac{R_1 R_2}{(2d)^2}}}{2\left(1 - \dfrac{R_1 R_2}{(2d)^2}\right)}, \\ \left(\omega_-^{\mathrm{th}}\right)^2 = \dfrac{\omega_{1_0}^2 + \omega_{2_0}^2 + \sqrt{\left(\omega_{1_0}^2 - \omega_{2_0}^2\right)^2 + 4\omega_{1_0}^2 \omega_{2_0}^2 \dfrac{R_1 R_2}{(2d)^2}}}{2\left(1 - \dfrac{R_1 R_2}{(2d)^2}\right)}. \end{cases} \tag{44}$$

We find the same expressions of the resonance frequencies as those given by Ida (2002) except that thermal effects are here taken into account through the frequencies ω_{1_0} and ω_{2_0} defined by Eq. (21). Note that Eq. (42) allows not only to calculate the frequencies but also the Q factors, which correspond to the ratio of the imaginary part of the complex resonance frequency on its real part.

IV. TWO BUBBLES RESONANCES

The resonances of two interacting identical bubbles are calculated for distances such as $1.01 \leq \beta = d/R \leq 10^3$, with $R_1 = R_2 = R = 100\ \mu$m. The resonances, which are complex numbers, are calculated numerically by searching for the zeros of the exact characteristic equation (36). The two resonances founded are denoted by $\omega_\pm^{\mathrm{num}}$. The discussion of the results makes use of the terminology introduced in Feuillade (1995), where interactive scattering by clusters of bubbles has been categorized in "symmetric modes," whereby bubbles oscillate in phase, and "antisymmetric modes," whereby bubbles oscillate in antiphase. The resonance frequencies given by $\mathrm{Re}\{\omega_\pm^{\mathrm{num}}\}$ are compared in Fig. 2(a) to those given by Eq. (44), and the Q factor of each resonance $\omega_\pm^{\mathrm{num}}$ is plotted in Fig. 2(b). The symmetric modes $\omega_+^{\mathrm{num,th}}$ typically show downward shifts in frequency and a tendency to increase damping and the antisymmetric modes $\omega_-^{\mathrm{num,th}}$ generally show upward frequency shifts and reduced damping. However, a new result can be observed if bubbles are very close, i.e., if $\beta < 2$, the antisymmetric mode frequency $\mathrm{Re}\{\omega_-^{\mathrm{num}}\}$ is very different from ω_-^{th}. This strong discrepancy can be explained by examining the number of modes used for numerical simulations. The dimension of the matrices in Eq. (36) is infinite, and the linear system of equations was numerically truncated to an order N, which is determined by applying the following convergence criterion:

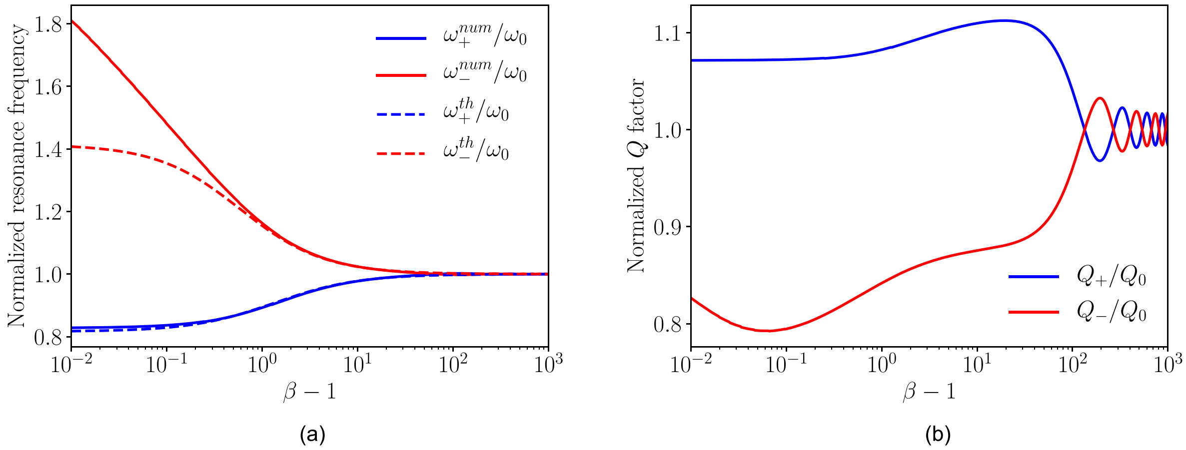

FIG. 2. (Color online) (a) Resonance frequencies and (b) Q factors as functions of the coefficient $\beta - 1$ for bubble radii $R_1 = R_2 = 100\ \mu$m.

$$\left| \frac{\omega_\pm^{\rm num}(N) - \omega_\pm^{\rm num}(N-1)}{\omega_\pm^{\rm num}(N-1)} \right| < 0.1\%. \tag{45}$$

The number N is plotted against the coefficient $\beta - 1$ in Fig. 3. This shows that it is necessary to use a large number of modes for very close bubbles and especially for the antisymmetric mode. As in Maksimov and Yusupov (2016), we can conclude that the traditional method taking into account only the monopole component in the interaction between the bubbles has a limited range of applicability and cannot be applied at distances compared with the size of the bubbles. A physical explanation is that antisymmetric modes are strongly dipolar (Feuillade, 1995), which requires using many modes to describe the scattered acoustic field around the two-bubble system.

Another interesting result is observed in Fig. 2(b). When the wavelength is close to the distance $2d$ between bubble centers, i.e., for $\beta \geq 50$, both resonance frequencies are very similar. But for $\beta \geq 100$, the Q factors oscillate in antiphase around the Q factor of a unique bubble. This kind of result was observed by Feuillade (2001) and can be explained as follows. The cyclic variation with separation occurs because the phase of the scattered field from one bubble alternately assists and retards the damping of the other as the distance between them changes.

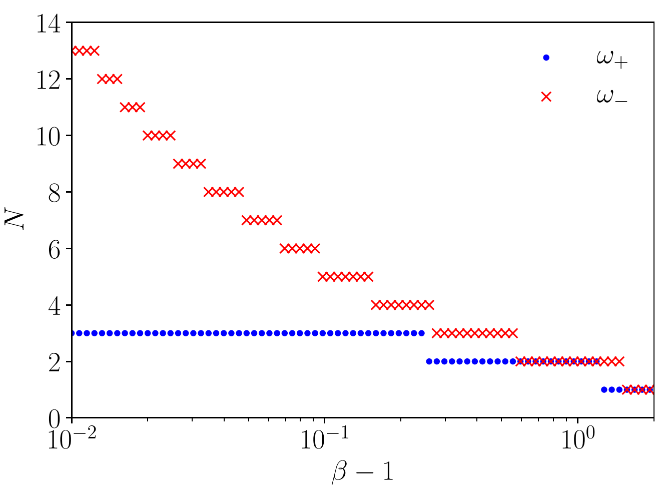

FIG. 3. (Color online) Number of modes to take into account for results of Fig. 2 plotted against coefficient $\beta - 1$.

The resonances are now calculated for two bubbles of different radii $R_1 = 100\ \mu$m and $R_2 = 150\ \mu$m with distances between them such as $1.01 \leq \beta \leq 10^3$. The same physical phenomena are observed concerning the resonance frequencies as shown in Fig. 4(a), but the behavior of the Q factors is very different [Fig. 4(b)]. In this case, the Q factors do not oscillate in or out of phase. The two bubbles no longer attract or repel each other because they do not oscillate at the same resonance frequency. Maybe each bubble is acting on the other one as an added mass effect. To conclude this section, it is worth noting that the assumption to not take into account the vorticity waves is valid as long as the viscous boundary layers of both bubbles do not intermingle [cf. Eq. (23)]. With the approximate value $\omega_\pm^{\rm num} R_1 \approx 25$, this assumption is respected for results shown in Fig. 2, for values of $\beta - 1$ superior to 0,02.

V. SCATTERING CROSS SECTIONS FOR THE TWO-BUBBLE SYSTEM

It is well known that bubbles are very efficient resonators in liquid, which can be highlighted by studying the scattering cross section. This parameter is especially used to show the different part of energy losses, i.e., thermal and viscous dissipations and acoustic radiation. In the context of a two-bubble system, the behavior of the scattering mechanisms is governed by the two resonances. The aim of this section is to derive the scattering cross section of the two-bubble system in order to determine the influence of both resonances.

The scattered field by both particles is given by

$$\varphi_s = \sum_{n=0}^{\infty} \left[B_n^{(1)} \chi_n(\mathbf{r}_1) + B_n^{(2)} \chi_n(\mathbf{r}_2) \right], \tag{46}$$

and using the addition theorem for $|\boldsymbol{r}_1|$ and $|\boldsymbol{r}_2|$ superior to d we get

$$\varphi_s(\boldsymbol{r}) = \sum_{\nu=0}^{\infty} C_\nu \chi_\nu(\mathbf{r}_0) \tag{47}$$

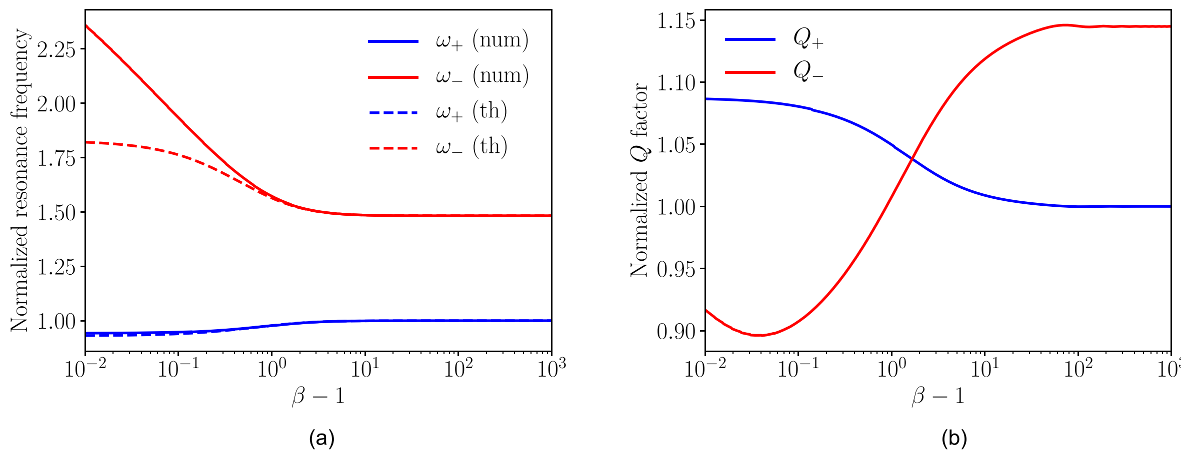

FIG. 4. (Color online) Same as Fig. 2 for bubble radii $R_1 = 100$ μm and $R_2 = 150$ μm.

with

$$C_\nu = \sum_{n=0}^{\infty}\sum_{\mu=0}^{\infty}\bar{G}(n,\nu,\mu)j_\mu(k_L d)\left[B_n^{(1)} + (-1)^\mu B_n^{(2)}\right]. \tag{48}$$

The differential scattering cross section of the two-bubble system is defined by

$$\sigma = \int_0^{2\pi}|f(\theta)|^2 d\theta \quad \text{with}$$
$$f(\theta) = \frac{1}{ik_L}\sum_{\nu=0}^{\infty}(-i)^\nu C_\nu P_\nu(\cos\theta). \tag{49}$$

Substituting Eq. (48) in Eq. (49) yields

$$\sigma = \frac{4\pi}{k_L^2}\sum_{n=0}^{\infty}\frac{|C_n|^2}{2n+1}. \tag{50}$$

Taking into account only the monopolar mode of each bubble and assuming that the two bubbles are identical, it comes from Eq. (38)

$$C_\nu = (2\nu+1)j_\nu(k_L d)\frac{\left[1+(-1)^\nu T_0^L h_0(2k_L d)\right]T_0^L}{1-\left(T_0^L\right)^2\left[h_0(2k_L d)\right]^2} \times \left[e^{jk_L d} + (-1)^\nu e^{-jk_L d}\right]. \tag{51}$$

It follows that

$$\begin{cases} C_\nu = 2(2\nu+1)j_\nu(k_L d)\dfrac{T_0^L\cos(k_L d)}{1-T_0^L h_0(2k_L d)}, & \text{if } \nu \text{ is even,} \\ C_\nu = 2i(2\nu+1)j_\nu(k_L d)\dfrac{T_0^L\sin(k_L d)}{1+T_0^L h_0(2k_L d)}, & \text{if } \nu \text{ is odd,} \end{cases} \tag{52}$$

and if the distance $2d$ between bubble centers is smaller than the wavelength, the effective scattering amplitudes of the monopolar and dipolar modes can be approximated by

$$\begin{cases} C_0 = -\dfrac{4ik_L d\omega^2}{(2\beta+1)\omega^2 + 4i\beta\omega\Gamma_p - 2\beta\omega_0^2}, \\ C_1 = \dfrac{4(k_L d)^3\omega^2}{(2\beta-1)\omega^2 + 4i\beta\omega\Gamma_p - 2\beta\omega_0^2}, \end{cases} \tag{53}$$

with $\omega_{10} = \omega_{20} = \omega_0$ the bubble Minnaert frequency. Equation (51) highlights that the monopolar mode $\nu = 0$ corresponds to bubbles that are in phase, while the dipolar mode $\nu = 1$ corresponds to bubbles in opposite phase. The scattering cross sections for $1.01 \le \beta \le 100$ are plotted as functions of the normalized frequency ω/ω_{1_0} in Fig. 5 for identical bubbles with $R_1 = R_2 = 100$ μm. For $\beta < 2$, i.e., for very close bubbles, the maximal value of scattering cross section normalized by the geometrical section $2S_1 = 8\pi R_1^2$ is slightly inferior to that of a unique bubble normalized by S_1, and for $2 \le \beta \le 50$, it is superior. Whatever the value of β, this scattering cross section has a maximum value at the frequency of the symmetric resonance, and the signature of the antisymmetric resonance of the two-bubble system is not visible on these curves. This result may be surprising, given the similar values of the Q factor of both resonances as

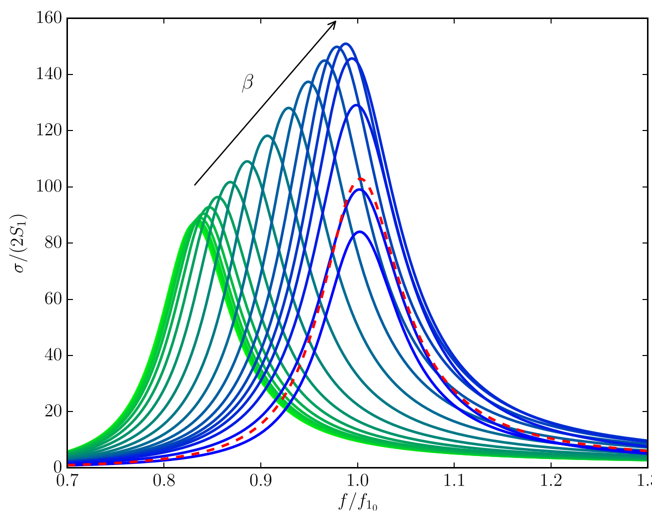

FIG. 5. (Color online) Scattering cross sections normalized by the geometrical section $4\pi R_1^2$ as functions of the normalized frequency ω/ω_{1_0}.

shown in Fig. 2(b). However, the amplitude C_1 of the dipolar mode associated to the two-bubble system is proportional to the term $(k_L d)^3$, while the amplitude C_0 of the monopolar mode is proportional to the term $k_L d$. This explains why for small values of the distance between bubbles compared to the wavelength, i.e., for $\beta < 2$, the acoustic radiation of the dipolar mode of the two-bubble system is weak compared to that of the monopolar mode.

In order to verify that symmetric and antisymmetric resonances are, respectively, monopolar and dipolar, the acoustic pressure scattered around the two bubbles is plotted in Fig. 6. It is clear that the symmetric resonance is predominantly monopolar with an acoustic field that looks like a breathing mode, while the antisymmetric mode is predominantly dipolar. Finally, this result supports the argument of Feuillade (1995) about super-resonances, which are associated with closely spaced bubbles oscillating in antiphase (Tolstoy, 1986). As they form dipoles, they are weakly coupled to the external fluid since the amplitude C_1 is small, and may therefore not be observed experimentally. However, for $2 \leq \beta \leq 50$, the acoustic radiation of the dipolar mode is not weak when the symmetric and antisymmetric resonance frequencies are getting closer. In this case, both effects associated to the monopolar and dipolar scattering are cumulative, which leads to observe a scattering cross section superior to that of a single bubble. Can we say that it is a super-resonance effect? We leave open the question, but what is clear is that the strengthening of the scattering observed close to the symmetric resonance frequency is due to the presence of the antisymmetric mode.

The scattering cross sections are now plotted in Fig. 7 for two different bubbles of radii $R_1 = 100$ μm and $R_2 = 150$ μm with $1.01 \leq \beta \leq 100$. Contrary to the previous case, we can see that both resonances have an influence on the scattering cross section. Even if the antisymmetric mode of bubble-bubble system is detected, the amplitude of the scattering cross section associated to the system remains always much smaller than for a single bubble. Clearly, that is not a favorable situation to observe a super-resonance effect. In fact,

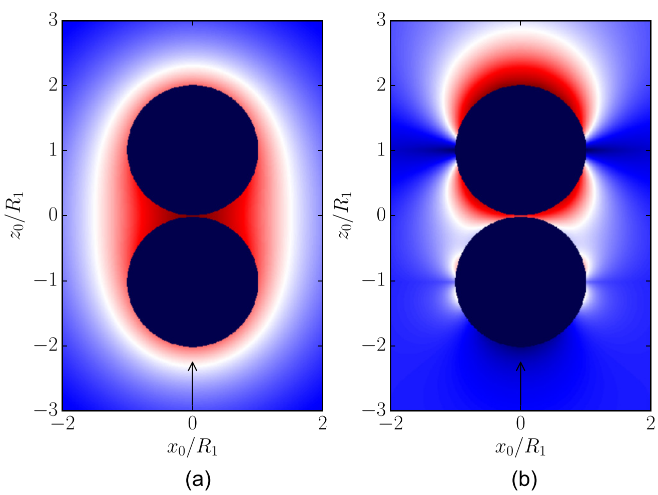

FIG. 6. (Color online) Acoustic pressure for $\beta = 1.01$ for symmetric and antisymmetric resonances. Arrows indicate the direction of propagation of the incident wave.

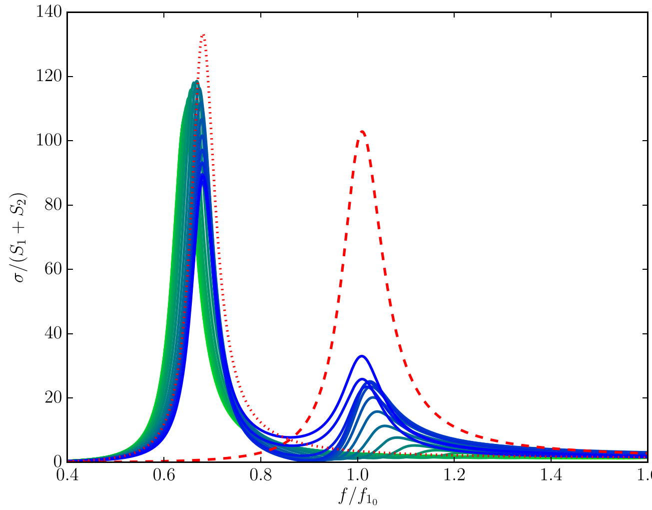

FIG. 7. (Color online) Scattering cross sections normalized by the geometrical section $S_1 + S_2 = 4\pi(R_1^2 + R_2^2)$ as functions of the frequency.

this confirms our intuition that the dynamic of the interaction between bubbles depends on whether the two bubbles are the same or not. Note to conclude that symmetric and antisymmetric modes have been detected experimentally in Hsiao *et al.* (2001). In this case, one might think that the two bubbles involved in the experimental setup were of different size.

VI. CONCLUSION

In this paper the mutual interaction between close bubbles has been modeled and analyzed. The resonances are usually categorized in symmetric modes, whereby bubbles oscillate in phase, and antisymmetric modes, whereby bubbles oscillate in antiphase. The analysis is developed in the linear framework of the multiple scattering theory, and it is based on the ECAH model of Epstein and Carhart (1953) and Allegra and Hawley (1972).

First of all, the ECAH model has been used to obtain the scattering coefficients for a unique bubble. Then, the scattering coefficient associated to the monopolar mode has been derived analytically in the long wavelength approximation. It has been shown that the ECAH model give the same results as those obtained by the hydrodynamic model based on the Rayleigh-Plesset equation, when the viscosity in the fluid and the thermal effects in the bubble are taken into account.

Using the multiple scattering theory, the exact characteristic equation has been obtained. The real part of the complex roots gives the resonance frequencies and their imaginary part the quality factors. Using the long wavelength approximation and restricting to monopolar modes of bubbles, the exact characteristic equation reduces to the one given by the coupled oscillator method (Ida, 2002). Resonance frequencies associated to symmetric modes are almost the same in both cases. This is not at all the case for antisymmetric modes for which taking a great number of vibration modes is necessary. The physical explanation of this strong discrepancy is clear. The symmetric mode is predominantly monopolar and is rather well represented by the vibration mode $n = 0$, while the antisymmetric mode is predominantly dipolar so that a great number of vibration modes are required to describe the scattered acoustic field around the two-bubble system. So, taking only into account the monopolar mode in the interaction between bubbles has a limited range of applicability when distances are comparable to the size of the bubbles.

For distances comparable to the size of the bubbles, the behavior of the quality factors is very different if the two bubbles are the same or not. When the wavelength is close to the distance between two identical bubbles, the Q factors oscillate in antiphase around the Q factor of one bubble. In this case, the cyclic variation with separation occurs because the phase of the scattered field from one bubble alternately assists and retards the damping of the other as the distance between them changes. This is not what is observed when bubbles are different. In this case, the Q factors do not oscillate in or out of phase. The two bubbles no longer attract or repel each other because they do not oscillate at the same resonance frequency. Maybe each bubble is acting on the other one as an added mass effect. Excepted for very close bubbles, we think that the dynamic of the interaction depends on whether the two bubbles are the same or not. To conclude, whatever the spacing between two identical bubbles, the scattering cross section has a maximum value at the frequency of the symmetric resonance, and the antisymmetric resonance is not detected. However, the strengthening of the scattering observed close to the symmetric resonance frequency is clearly due to the presence of the antisymmetric mode.

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